Challenge problem 2011-2012 #02

Solution proposed by Philippe Fondanaiche, Paris ,France

Let k^2, $(k+1)^2$,.....$(k+n-1)^2$ the n consecutive perfect squares with k and n integers such that k >0 and n >1

So we have the relation $(k^2 + (k+1)^2 + ... + (k+n-1)^2) / n = n^2$

By developing it, we get $(nk^2 + 1^2 + 2^2 + ... + (n-1)^2 + 2k(1 + 2 + ... + n-1))/n = n^2$,then the quadratic equation in k: $6k^2 + 6(n-1)k - 4n^2 - 3n + 1 = 0$

As k is an integer, the discriminant is a perfect square p^2 of an integer $p > 0$.

So $9(n-1)^2 + 6(4n^2 + 3n - 1) = p^2$ which gives the diophantine equation $33n^2 + 3 = p^2$

p is a multiple of 3 . Let $p = 3q$. So $11n^2 + 1 = 3q^2$ and $k = (q - n + 1)/2$

There is a trivial solution $n(0) = 1$ and $q(0) = 2$ which is excluded as $n > 1$.

The general solution $[n(i),q(i)]$ is given the reccurence relations:

$n(i+1) = 23n(i) + 12q(i)$ and $q(i+1) = 44n(i) + 23q(i)$

Then $k(i) = (21n(i) + 11y(i) + 1)/2$

The following sheet provides the first values of n and k meeting the conditions of the problem.

n	k
47	22
2 161	989
99 359	45 450
4 568 353	2 089 689
210 044 879	96 080 222
9 657 496 081	4 417 600 501
444 034 774 847	203 113 542 802
20 415 942 146 881	9 338 805 368 369
938 689 303 981 679	429 381 933 402 150