The Evaluation of $\int_0^1 \lfloor -\ln x \rfloor dx$.

We consider the more general problem:

$$\int_0^1 \lfloor -\ln x \rfloor^k \, dx.$$

where k is a positive integer.

Now, if n is a nonnegative integer,

$$\begin{aligned} &e^{-n} \ge x \ge e^{-n-1} \\ &-n \ge \ln x \ge -n-1 . \\ &n \le -\ln x \le n+1 \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 \lfloor -\ln x \rfloor^k \, dx &= \sum_{n=0}^{+\infty} \int_{e^{-n-1}}^{e^{-n}} \lfloor -\ln x \rfloor^k \, dx = \sum_{n=0}^{+\infty} \int_{e^{-n-1}}^{e^{-n}} n^k \, dx \\ &= \sum_{n=0}^{+\infty} \left(e^{-n} - e^{-n-1}\right) n^k = \left(1 - e^{-1}\right) \sum_{n=0}^{+\infty} n^k e^{-n} . \end{aligned}$$

Now, for $|x| < 1$, we have

$$\sum_{n=0}^{+\infty} n^k x^n = \frac{P_k(x)}{(1-x)^{k+1}},$$

where $P_k(x)$ is a polynomial of degree k. In particular,

$$\sum_{n=0}^{+\infty} n x^n = \frac{x}{(1-x)^2} \text{ and } \sum_{n=0}^{+\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

Thus

$$\int_0^1 \lfloor -\ln x \rfloor dx = \left(1 - e^{-1}\right) \sum_{n=0}^{+\infty} n e^{-n} = \left(1 - e^{-1}\right) \frac{e^{-1}}{\left(1 - e^{-1}\right)^2} = \frac{1}{e-1}$$

and

$$\int_0^1 \lfloor -\ln x \rfloor^2 dx = \left(1 - e^{-1}\right) \sum_{n=0}^{+\infty} n^2 e^{-n} = \left(1 - e^{-1}\right) \frac{e^{-1}\left(1 + e^{-1}\right)}{\left(1 - e^{-1}\right)^3} = \frac{e+1}{(e-1)^2} .$$

One can obtain similar results for $k \ge 3$. For example,

$$k = 3: \text{ integral} = \frac{e^2 + 4e + 1}{(e-1)^3}$$

$$k = 4: \text{ integral} = \frac{e^3 + 11e^2 + 11e + 1}{(e-1)^4}$$

$$k = 5: \text{ integral} = \frac{e^4 + 26e^3 + 66e^2 + 26e + 1}{(e-1)^5}.$$

One might also note that if $k = 0$, then we have, as above,

$$\int_0^1 dx = \left(1 - e^{-1}\right)\sum_{n=0}^{+\infty} e^{-n} = \left(1 - e^{-1}\right)\frac{1}{1 - e^{-1}} = 1.$$

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