OC458. Let A be the product of eight consecutive positive integers and let k be the largest positive integer for which $k^4 \leq A$. Find the number k knowing that it is represented in the form $2p^m$, where p is a prime number and m is a positive integer.

Originally Problem 4, Grade 9 of the 2017 Bulgaria Math Olympiad.

We received 6 submissions of which 4 were complete. We present the solution based on the submissions of UCLan Cyprus Problem Solving Group and Problem Solving Group of Missouri State University (done independently).

First, we claim that if $A = (n+1)(n+2)\cdots(n+8)$ for some integer number $n \geq 0$, then $k = (n+2)(n+7)$. To establish this we use the following inequalities

$$(x+\sqrt{yz})^2 \leqslant (x+y)(x+z) \leqslant \left(x+\frac{y+z}{2}\right)^2$$

that are valid for x, y, z positive real numbers. Both inequalities are equalities if and only if $y = z$. The inequalities follow easily from AM-GM inequality.

We write

$$A = (n^2+9n+8)(n^2+9n+14)(n^2+9n+18)(n^2+9n+20).$$

Then

$$(n^2+9n+\sqrt{8\times 20})^2(n^2+9n+\sqrt{14\times 18})^2 < A$$

and

$$A < \left(n^2+9n+\frac{8+20}{2}\right)^2\left(n^2+9n+\frac{14+18}{2}\right)^2,$$

so

$$(n^2+9n+\sqrt[4]{8\times 14\times 18\times 20})^2 < A < \left(n^2+9n+\frac{8+14+18+20}{2}\right)^2.$$

In conclusion,

$$(n^2+9n+14)^2 < A < (n^2+9n+15)^2,$$

so we must have

$$k = n^2+9n+14 = (n+2)(n+7).$$

If $(n+2)(n+7) = 2p^m$ with p prime and $m > 0$, then the factors $n+2$ and $n+7$ can be $2p^a$ or p^b. If a and b are both positive, then p divides the factor difference: $(n+7)-(n+2) = 5$. So we must have $p = 5$, $n+7 = 2\times 5$, and $n+2 = 5$, leading to $k = 50$. If $a = 0$, then $n+2 = 2$, and $n+7 = 7$. So we must have $k = 2\times 7 = 14$, a number of the required form. If $b = 0$, then $n+2 = 1$ giving $n = -1$, leading to no valid solution as $n > 0$.

Thus the only possible values for k are $k = 14 = 2\times 7$ and $k = 50 = 2\times 5^2$.