

A2826: A Calculation Error

Zig chose twenty positive real numbers x_i , $1 \leq i \leq 20$, whose sum is equal to 85 and the sum of their inverses is equal to 24. He then calculates

$$S = \sum_{\substack{1 \leq x_i \leq 20 \\ 1 \leq x_j \leq 20 \\ i \neq j}} \frac{x_i}{x_j}.$$

Puce for his part chose a sequence of k positive real numbers y_i , $1 \leq i \leq k$ whose sum is equal to 159 and the sum of their inverses is equal to 13. He asserts that

$$T = \sum_{\substack{1 \leq y_i \leq k \\ 1 \leq y_j \leq k \\ i \neq j}} \frac{y_i}{y_j}$$

is also equal to S .

Determine S and show that Puce made a calculation error.

A4919: One algebraic equation and two Diophantine ones

Q1: Find all real solutions to the equation $18^x - 8^x = 27^x - 18^x$.

Q2: Find all positive integer solutions to the equation $x^2 + 26455 = 2^y$.

Q3: For a certain positive integer x , $10 + x$ and $4000 + x$ are respectively the first and seventh terms of a sequence of integers forming a geometric progression whose ratio is a rational number. Determine the fourth term in the sequence.

A817: An Unusual Calculator

Zig just received a calculator for his birthday. The calculator only has three keys. Given any positive integer n it will compute:

- 1) $\varphi(n)$, Euler's function, the number of integers which are strictly less than the integer n and are relatively prime to it.
- 2) $\sigma(n)$ the sum of the divisors of the integer n , including 1 and itself.
- 3) $\tau(n)$ the number of divisors of the integer n , including 1 and itself.

Q1: The integer 7 is displayed on the screen. Help Zig get the integer 5 by pressing some sequence involving these three keys. [★★]

Q2: Starting with the integer 2 and operating as before, help Zig to successively obtain all the integers from 3 to 25 (not necessarily in this order). [★★★★]

Q3: For the more courageous: Given two distinct integers p and q greater than 1. Prove that Zig can always go from p to q in a finite number of steps using these three keys. [★★★★★]

D2923: A Garden of Mathematicians

A gardener designed a marvelous garden in which he installed the statues of famous mathematicians and geometers identified by the first capital letter of their name.

He begins by tracing the outlines of a circular flower bed Ω with center O (Ocagne), inside which he draws two chords GI (Gauss - Isidore de Milet) and HJ (Hipparque - Julia) perpendicular to each other at a point Z (Zeno of Elea) distinct from O.

Secondly, he traces the paths of a complete quadrangle ABCDEF in memory of Archimedes, Brocard, Chasles, Descartes, Euler and Feuerbach. The aisles AB, BC, CD, DA are respectively tangent at G, H, I, J to the circle (Ω). The statues of Euler and Feuerbach are respectively at the intersection of lines AD and BC on the one hand, AB and CD on the other.

He then draws in this order the midpoints K, L, M and N of the segments AC, BD, GI and HJ in order to install the statues of Klein, Lemoine, Monge and Newton.

It only remains for him to mark the point P center of the circular lawn (ω) circumscribed around the triangle ABC in order to put the statue of Pascal there and the point T reserved for the statue of Thales at the intersection of the right EF with the line OZ.

Prove that:

- 1) the lines EO and FO,
 - 2) the lines OT and EF,
 - 3) the lines PZ and EF,
 - 4) the lines EP and FZ,
 - 5) the lines FP and EZ,
 - 6) the lines KL and MN,
- are pairwise perpendicular.

E 139: Iterated Indices

The positive integers $a_1 = 12, a_2, a_3, \dots$ form an arithmetic progression with a difference greater than 1. There exists an integer k such that

$$a_{a_k} = 2000.$$

Find

$$a_{a_{a_k}}.$$

G1911: Randomly Sliced Pizza

The pizzas at a pizzeria are circular in shape and are cut by a robot. Each time a slice is made two points are chosen randomly and independently of one another along the circumference of the pizza (uniformly) and then a cut is made along the chord that connects these two points.

Q1: You request a medium sized pizza specifying that the robot makes exactly three cuts. Determine the expected number of pieces you will be served.

Q2: You want a very large pizza for 15 people. Determine the number of cuts you want to ask the robot to get a math expectation as close to 15 pieces as possible.

Note: The pieces will not, in general, have the same size.

Problems from Crux MathemAttic, due December 15, 2020

MA86. On a $2 \times n$ board, you start from the square at the bottom left corner. You are allowed to move from square to adjacent square, with no diagonal moves, and each square must be visited at most once. Moreover, two squares visited on the path may not share a common edge unless you move directly from one of them to the other. We consider two types of paths, those ending on the square at the top right corner and those ending on the square at the bottom right corner. The diagram below shows that there are 4 paths of each type when $n = 4$. Prove that the numbers of these two types of paths are the same for $n = 2014$.



MA87. One or more pieces of clothing are hanging on a clothesline. Each piece of clothing is held up by either 1, 2 or 3 clothespins. Let a_1 denote the number of clothespins holding up the first piece of clothing, a_2 the number of clothespins holding up the second piece of clothing, and so forth. You want to remove all the clothing from the line, obeying the following rules:

- (i) you must remove the clothing in the order that they are hanging on the line;
- (ii) you must remove either 2, 3 or 4 clothespins at a time, no more, no less;
- (iii) all the pins holding up a piece of clothing must be removed at the same time.

Find all sequences a_1, a_2, \dots, a_n of any length for which all the clothing can be removed from the line.

MA88. *Proposed by Konstantin Knop.*

- a) Sort the numbers from 1 to 100 in increasing order of their digit-sums; in case of a tie, sort in increasing order of the numbers themselves. Consider the resulting sequence $a(n)$: $a(1) = 1, a(2) = 10, a(3) = 100, \dots$. Find at least one number $n > 1$ such that $a(n) = n$.
- b) Consider the same problem but for numbers from 1 to 100 000 000.

MA89. *Proposed by Bill Sands.*

Two robots R2 and D2 are at the origin O on the x, y plane. R2 can move twice as fast as D2. There are two treasures located on the plane, and whichever robot gets to each treasure first gets to keep it (in case of a tie, neither robot gets the treasure). One treasure is located at the point $P = (-3, 0)$, and the other treasure is located at a point $X = (x, y)$. Find all $X \neq O$ so that D2 can prevent R2 from getting both treasures, no matter what R2 does. Which such X has the largest value of y ?

Note: D2 does not care if R2 gets one of the treasures, only that R2 shouldn't get both treasures. D2 also doesn't care if it gets either treasure itself, it only wants to prevent R2 from getting both treasures.

MA90. *Proposed by Michel Bataille.*

Two positive integers are called co-prime if they share no common divisors other than 1. Find all pairs of co-prime x, y such that $\frac{y(x+y)}{x-y}$ is a positive integer.

Problems from Crux Olympiad Corner, due December 15, 2020

OC496. The six digits 1, 2, 3, 4, 5, and 6 are used to construct a one-digit number, a two-digit number and a three-digit number. Each digit must be used only once and all six digits must be used. The sum of the one-digit number and the two-digit number is 47 and the sum of the two-digit number and the three-digit number is 358. Find the sum of all three numbers.

OC497. Does there exist a positive integer that is divisible by 2020 and has equal number of digits 0, 1, 2, ..., 9?

OC498. A collection of 8 cubes consists of one cube with edge-length k for each integer k , $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

- (a) Any cube may be the bottom cube in the tower.
- (b) The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

OC499. A self-avoiding rook walk on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook's path is non-self-intersecting.

Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chess board which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m ; $R(2, 2) = 2$; $R(3, 2) = 4$; $R(3, 3) = 11$. Find a formula for $R(3, n)$ for each natural number n .

OC500. An $n \times m$ matrix is *nice* if it contains every integer from 1 to mn exactly once and 1 is the only entry which is the smallest both in its row and in its column. Prove that the number of $n \times m$ nice matrices is $\frac{(nm)!n!m!}{(n+m-1)!}$.

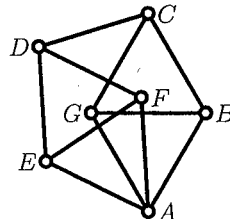
Problems from Crux Mathematicorum, due December 15, 2020

4571. *Proposed by Ed Barbeau. Dedicated in memoriam to Richard K. Guy.*

What is the smallest square integer expressible as the product of three distinct nonzero integers in arithmetic progression?

4572. *Proposed by Veselin Jungić. Dedicated in memoriam to Richard K. Guy.*

In 1961, Canadian mathematicians Leo and William Moser introduced a geometric object consisting of seven vertices and eleven line segments of the unit length. This object is now known as the Moser spindle: see p. 390-396 of this issue for more details.



In the Moser spindle, find the measure of the angle $\angle GAF$.

4573. *Proposed by J. Chris Fisher.*

For any triangle ABC let γ be the circle through A and B that surrounds the incircle α and is tangent to it, while β is a circle inside the triangle that is tangent to the sides AC and BC . Then β is externally tangent to γ if and only if it is also tangent to the line parallel to (but not equal to) AB that is tangent to the incircle.

This result was conjectured following the solution of Honsberger problem H4 [2018: 143-144], which related H4 to Problem 2.6.4 in H. Fukagawa and D. Pedoe, Japanese Temple Geometry Problems: San Gaku, The Charles Babbage Research Centre (1989) page 37.

4574. *Proposed by George Apostolopoulos.*

Let x_1, \dots, x_n be positive real numbers with $x_i < 64$ such that $\sum_{i=1}^n x_i = 16n$. Prove that

$$\sum_{i=1}^n \frac{1}{8 - \sqrt{x_i}} \geq \frac{n}{4}.$$

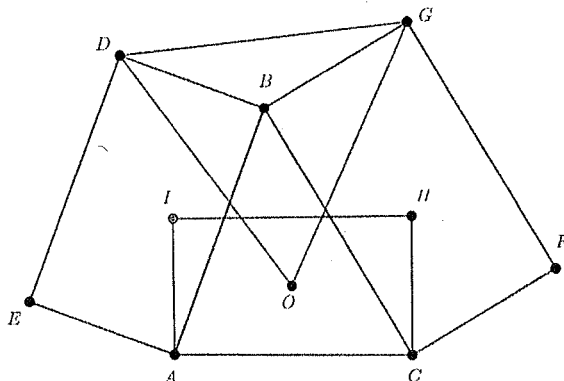
4575. *Proposed by Nguyen Viet Hung.*

Determine the coefficient of x in the following polynomial

$$\left(1 + \binom{n}{0}x\right) \left(1 + \binom{n}{1}x\right)^2 \left(1 + \binom{n}{2}x\right)^3 \cdots \left(1 + \binom{n}{n}x\right)^{n+1}.$$

4576. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABDE$, $BCFG$ and $ACHI$ be three similar rectangles as given in the figure. Suppose $\frac{AB}{AE}$ is constant and let O be the center of $ACHI$. Show that $OD = OG$ and $\angle GOD$ is constant when A and C are fixed but B can move.



4577. *Proposed by Nikolai Osipov.*

For any integer k , solve the equation

$$xy^2 + (kx^2 + 1)y + x^4 + 1 = 0$$

in integers x, y .

4578. *Proposed by Ed Barbeau. Dedicated in memoriam to Richard K. Guy.*

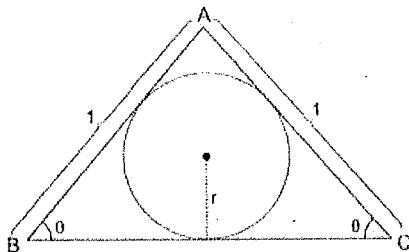
Suppose that $\{a, b, c\}$ and $\{u, v, w\}$ are two distinct sets of three integers for which $a + b + c = u + v + w$ and $a^2 + b^2 + c^2 = u^2 + v^2 + w^2$. What is the minimum possible value assumed by $|abc - uvw|$?

4579. *Proposed by George Stoica.*

Let $a, b, c \in \mathbb{Z}^*$ such that $\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \in \mathbb{Z}$. Prove that $\frac{ab}{c}, \frac{bc}{a}, \frac{ca}{b} \in \mathbb{Z}$.

4580. *Proposed by Alpaslan Ceran.*

In an isosceles triangle ABC with $AB = AC = 1$, find the length of BC which maximizes the inradius.



We reprint problem 12185 from the May 2020 issue, correcting errors.

12185. *Proposed by George Stoica, Saint John, NB, Canada.* Let n_1, \dots, n_k be pairwise relatively prime odd integers greater than 1. For $i \in \{1, \dots, k\}$, let $f_i(x) = \sum_{m=1}^{n_i} x^{m-1}$. Let A be a $2k$ -by- $2k$ matrix with real entries such that $\det f_j(A) = 0$ for all $j \in \{1, \dots, k\}$. Prove $\det A = 1$.

12195. *Proposed by Joseph DeVincentis, Salem, MA, James Tilley, Bedford Corners, NY, and Stan Wagon, Macalester College, St. Paul, MN.* For which integers n with $n \geq 3$ can a regular n -gon be inscribed in a cube? The vertices of the n -gon must all lie on the cube but may not all lie on a single face.

12196. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.* Determine which positive integers n have the following property: If a_1, \dots, a_n are n real numbers greater than or equal to 1, and A , G , and H are their arithmetic mean, geometric mean, and harmonic mean, respectively, then

$$G - H \geq \frac{1}{G} - \frac{1}{A}.$$

12197. *Proposed by Nicolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Prove that the equation

$$(a^2 + 1)(b^2 - 1) = c^2 + 3333$$

has no solutions in integers a , b , and c .

12198. *Proposed by Michel Bataille, Rouen, France.* Let $A_1A_2A_3$ be a nonequilateral triangle with incenter I , circumcenter O , and circumradius R . For $i \in \{1, 2, 3\}$, let B_i be the point of tangency of the incircle of $A_1A_2A_3$ with the side of the triangle opposite A_i , and let C_i be the point of intersection between the circle centered at I of radius R and the ray IB_i . Let K be the orthocenter of $C_1C_2C_3$. Prove that I is the midpoint of OK .

12199. *Proposed by Shivam Sharma, Delhi University, New Delhi, India.* Prove

$$\int_0^\infty \frac{x \sinh(x)}{3 + 4 \sinh^2(x)} dx = \frac{\pi^2}{24}.$$

12200. *Proposed by Ibrahim Suat Evren, Denizli, Turkey.* Prove that for every positive integer m , there is a positive integer k such that k does not divide $m + x^2 + y^2$ for any positive integers x and y .

12201. *Proposed by Stephen M. Gagola, Jr., Kent State University, Kent, OH.* Let F be a field, and let G be a finite group. The group algebra $F[G]$ is the vector space of all formal sums $\sum_{g \in G} a_g g$, where $a_g \in F$, with multiplication defined by extending the multiplication in G via the distributive laws. A subset S of $F[G]$ is G -invariant if $s \in S$ and $g \in G$ imply $sg \in S$. In particular, the subset G is G -invariant, as is the singleton set $\{\sum_{g \in G} g\}$. Find all fields F and groups G such that there exists an F -linear transformation $\phi: F[G] \rightarrow F[G]$ that is not right multiplication by an element of G but that nevertheless sends every G -invariant subset to itself.