Recursive Sequences and Groups

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A look inside Recursive Sequences and Groups

Outline

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1. Characteristic Polynomial
2. Examples
3. Results
4. Some open questions

Outline

The Fibonacci Sequence and its characteristic polynomial

$$f_n = f_{n-1} + f_{n-2}$$
$$0, 1, 1, 2, 3, 4, 8, 13, \ldots.$$

$$r^n = f_n$$

- $r^n = r^{n-1} + r^{n-2}$
- $r^2 = r + 1$
- $r^2 - r - 1 = 0$

$p(x) = x^2 - x - 1$

Using the Quadratic Formula:

- $x_1 = (1+\sqrt{5})/2$
- $x_2 = (1-\sqrt{5})/2$
- $f_n = \alpha x_1^n + \beta x_2^n$
- $\alpha + \beta = 0$
- $\alpha(1+\sqrt{5})/2 + \beta(1-\sqrt{5})/2 = 1$
- $f_n = (1/\sqrt{5})((1+\sqrt{5})/2)^n - (1/\sqrt{5})((1-\sqrt{5})/2)^n$

Fibonacci and the group Z_3

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \ldots.$$

- Period is 8
- $x^2 - x - 1 \mid x^d - 1$
- $(x^8 - 1) = (x^2 - x - 1)(x^6 + x^5 + 2x^4 + 2x^2 + 2x + 1)$
- So the coefficients determine the characteristic polynomial, which in turn determines the period.

Coefficients and the fibonacci recursion

$$f_n = 2f_{n-1} + f_{n-2}$$
$$p(x) = x^2 - 2x - 1$$

What does this mean in a group operation?

- $\psi : Z_m \longrightarrow Z_m$, given by:

  $\psi(x) = 2x$ is a homomorphism
- Try using group homomorphisms rather than exponents for the coefficients

Using homomorphisms as coefficients

- $Z_n = \langle x \rangle$ and $Z_2 = \langle y \rangle$
- $D_{2n} \cong Z_n \rtimes Z_2$
- $\varphi : Z_2 \longrightarrow Aut(Z_n)$
- $\varphi(y)(h) = h^{-1}$
- $\psi_i : D_{2n} \rightarrow D_{2n}$
- $f_n = \psi_i(f_{n-1})f_{n-2}$
- $\psi_i((a, b)) = (x^i, 1)(a, b)(x^{-i}, 1)$

Using $f_0 = (x, 1)$ and $f_1 = (1, y)$, as our initial terms, the first six terms of our sequence are:

$$(x, 1), (1, y), (x^{2i-1}, y), (x^{4i-1}, 1), (x^{6i-2}, y), (x^{4i-1}, y)$$

.

And the next term?

$$
\begin{aligned}
f_{6k} &= (x, 1) \\
f_{6k+1} &= (x^{4ki}, y) \\
f_{6k+2} &= (x^{(4k+2)i-1}, y) \\
f_{6k+3} &= (x^{4i-1}, 1) \\
f_{6k+4} &= (x^{(4k+6)i-2}, y) \\
f_{6k+5} &= (x^{(4k+4)i-1}, y)
\end{aligned}
$$

Period is $6k$ or $6k + 3$, for the smallest k

Finding the Period

If $n \mid 4i - 2$ and $n \mid (4k+6)i - 2$,
then we easily see that the period is $6k + 3$. But:

$$(4k+6)i - 2 = 4ki + 6i - 2 = (4ki + 2i) + (4i - 2)$$

so it is obvious that n need only divide $2i(2k+1)$. However, if n is odd, then $n \mid (2k+1)$ is sufficient. This is due to the fact that if $n \mid 4i - 2$, where n is odd, we actually have $n \mid 2i - 1$. So if $n \mid 2i - 1$, then $n \not| 2i$. Furthermore, if n does not satisfy the above case, then the period is $6k$, for the smallest k such that $4ki = ln$, where $l \in Z$ and n and i are given.

Initial Conditions:

$$f_0 = (x, 1), f_1 = (1, y) \ ; \ f_0 = (1, y), f_1 = (x, 1)$$

$$f_0 = (1, y) = f_1$$

- $f_n = f_{n-2}\psi_i(f_{n-1})$
- $f_n = f_{n-1}\psi_i(f_{n-2})$
- $f_n = \psi_i(f_{n-2})f_{n-1}$
- $f_n = \psi_i(f_{n-1})f_{n-2}$

Initial Conditions:

$$f_0 = (x, 1), f_1 = (1, y) \ ; \ f_0 = (1, y), f_1 = (x, 1)$$
$$f_0 = (1, y) = f_1$$

1. $\psi_i((a, b)) = (x^i, 1)(a, b)(x^{-i}, 1)$
2. $\psi_j((a, b)) = (x^j, 1)(a, b)(x^{-j}, 1)$

- $f_n = \psi_j(f_{n-2})\psi_i(f_{n-1})$
- $f_n = \psi_j(f_{n-1})\psi_i(f_{n-2})$
- $f_n = \psi_i(f_{n-2})\psi_j(f_{n-1})$
- $f_n = \psi_i(f_{n-1})\psi_j(f_{n-2})$

When $i = j$

- Period is easily found
- $f_n = \psi_j(f_{n-2})\psi_i(f_{n-1})$
  $= \psi_i(f_{n-2})\psi_i(f_{n-1}) = \psi_i(f_{n-2}f_{n-1})$
- $f_n = \psi_j(f_{n-1})\psi_i(f_{n-2})$
  $= \psi_i(f_{n-1})\psi_i(f_{n-2}) = \psi_i(f_{n-1}f_{n-2})$
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- $f_n = \psi_i(f_{n-1})\psi_j(f_{n-2})$
  $= f_n = \psi_i(f_{n-1})\psi_i(f_{n-2}) = \psi_i(f_{n-1}f_{n-2})$

Example?

Check It Out!

$$f_0 = (x, 1), f_1 = (1, y)$$

When $i = j$:

$$\begin{aligned}
f_n &= \psi_i(f_{n-1})\psi_j(f_{n-2}) = \psi_i(f_{n-1}f_{n-2}) \\
f_{6k} &= (x, 1) \\
f_{6k+1} &= (x^{4k(i+j)}, y) = (x^{8ki}, y) \\
f_{6k+2} &= (x^{4k(i+j)+2i-1}, y) = (x^{8ki-2i-1}, y) \\
f_{6k+3} &= (x^{4i-2j-1}, 1) = (x^{2i-1}, 1) \\
f_{6k+4} &= (x^{4k(i+j)+6i-2}, y) = (x^{8ki+6i-2}, y) \\
f_{6k+5} &= (x^{4k(i+j)+4i+2j-1}, y) = (x^{8ki+6i-1}, y)
\end{aligned}$$

PERIOD?

When $i \neq j$

- Period can be found in the exact some way
- More variables means trouble
- Simplier means would be nice

What If?

1. What if φ was defined differently?
   - How would redefining φ affect the sequence?
2. What if ψ was defined differently?
   - How would other homomorphisms affect the sequence?
3. What if the degree of our sequence was three or four, instead of two?
4. What if we use another group?