Genera of Zero-divisor Graphs of Finite Rings

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Graphs

Definition

A *graph* is a set of vertices along with a set of edges. Each edge connects two vertices. Two vertices are said to be *adjacent* if there is an edge connecting them.

Definition

H is a *subgraph* of G if H is a graph whose vertex set is a subset of G's vertex set, and whose edge set is a subset of G's edge set.

Graph examples

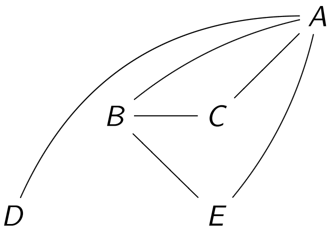

Figure: A graph

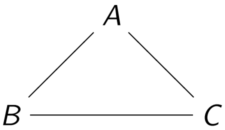

Figure: A subgraph of the previous graph

Complete graphs

Definition

A *complete graph* is a graph in which all vertices are adjacent. We denote the complete graph with n vertices by K_n.

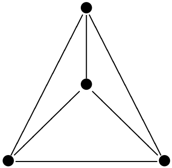

Figure: K_4

Complete bipartite graphs

Definition

A *complete bipartite graph* consists of two sets of vertices with all vertices in one set adjacent to all vertices in the other set, and no two vertices in the same set adjacent. We denote the complete bipartite graph where the first vertex set has n elements and the second vertex set has m elements by $K_{n,m}$.

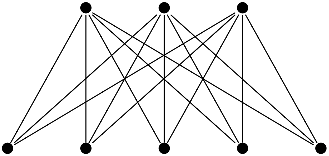

Figure: $K_{3,5}$

Planar graphs

Definition

A *planar graph* is a graph that can be drawn on a plane without any edges crossing.

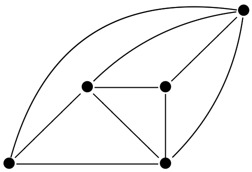

Figure: This graph is planar

Surfaces of crosscap k

The boundary of a Möbius band is a circle. A crosscap can be added to a surface by removing a disc and connecting a Möbius band by identifying the points on the boundary of the band with the points on the boundary of the removed disc. The sphere has crosscap 0, and the surface of crosscap k may be formed by adding k crosscaps to the sphere. The surface of crosscap 1 is called the *projective plane*. The surface of crosscap 2 is called the *Klein bottle*.

Fundamental Polygons

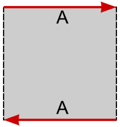

Figure: Fundamental polygon of the Möbius band

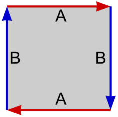

Figure: Fundamental polygon of the projective plane

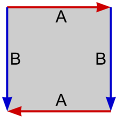

Figure: Fundamental polygon of the Klein bottle

Genus of a graph

Definition

A graph is said to have *genus* g if it can be drawn on a surface of genus g with no edges crossing, but cannot be drawn on a surface of a lesser genus with no edges crossing.

- The genus of the complete graph of n vertices is $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.
- The genus of the complete bipartite graph whose first vertex set has n elements and whose second vertex set has m elements is $\gamma(K_{n,m}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$.
- The genus of a graph is greater than or equal to the genus of any of its subgraphs.

Non-orientable genus

Definition

A graph is said to have *non-orientable genus* or *crosscap k* if it can be drawn on a surface of crosscap k with no edges crossing, but cannot be drawn on a surface of a lesser crosscap with no edges crossing.

- The non-orientable genus of the complete graph of n vertices is $\bar{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$, with the exception that $\bar{\gamma}(K_7) = 3$.
- The non-orientable genus of the complete bipartite graph whose first vertex set has n elements and whose second vertex set has m elements is $\bar{\gamma}(K_{n,m}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil$.
- The non-orientable genus of a graph is greater than or equal to the non-orientable genus of any of its subgraphs.

Rings

We can think of a *ring* as a set in which we can add and multiply elements, multiplication distributes over addition, and we always assume that addition is commutative and there is an additive identity, 0. If the multiplication operation is *commutative*, then we say that the ring is commutative. If there is a *multiplicative identity*, then we say that the ring has unity. We will only be discussing commutative rings with unity.

Zero-divisors

Definition

An element $a \in R$ is called a *zero-divisor* if there exists a nonzero b such that $ab = 0$.

Example

In the ring $\mathbb{Z}_8$, of the integers modulo 8, 6 is a zero-divisor because $6 \cdot 4 = 24 \equiv 0$.

Zero-divisor graph

Definition

The *zero-divisor graph* of a ring R, denoted $\Gamma(R)$, is the graph formed by taking all of the zero-divisors, except 0, of R as vertices and connecting two vertices iff their product is zero.

$$2 \text{ --- } 3 \text{ --- } 4$$

Figure: The zero-divisor graph of $\mathbb{Z}_6$

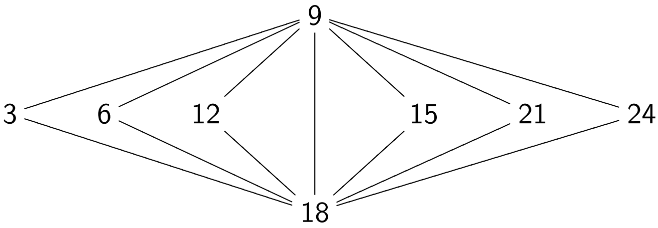

Figure: The zero-divisor graph of $\mathbb{Z}_{27}$

Local rings

Definition

- A subset $I \subseteq R$ is called an *ideal* if it is nonempty, closed under addition and additive inverses, and if for any $a \in I$ and $b \in R$, ab is in I.
- A *proper ideal* of a ring is an ideal which is not the entire ring.
- A *maximal ideal* is a proper ideal which is not contained in any other proper ideal.
- A *local ring* is a ring which has only one maximal ideal.

What we've been up to

We have been attempting to classify all finite rings with zero-divisor graphs of crosscap 2.

Methodology

- The *residue field* of a local ring R is the field obtained by modding out by the maximal ideal m.
- If I is an ideal and n is a natural number, then I^n is the ideal formed by taking all finite sums of products of n elements of I.
- The *index of nilpotency* of a local ring R with maximal ideal m is the least natural number n such that m^n is the zero ideal.
- In a finite local ring, all of the elements of the maximal ideal are zero-divisors and all of the elements not in the maximal ideal are units (i.e. they have a multiplicative inverse).

There are no local rings of crosscap 2

- If the index of nilpotency of a local ring is 2, then all products of elements of the maximal ideal are zero. Therefore, the zero-divisor graph is complete, and so since there are no complete graphs of crosscap 2, we only have to consider the case where the index of nilpotency is greater than 2.

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- If the residue field has order of at least 4, and the index of nilpotency is greater than 2, then the crosscap of the zero-divisor graph is greater than 4.
- We have gone through the rings of residue field of order 2 or 3 and small index of nilpotency (the index of nilpotency provides a lower bound on the crosscap), and found that there aren't any of crosscap 2.

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- We have gone through the rings of residue field of order 2 or 3 and small index of nilpotency (the index of nilpotency provides a lower bound on the crosscap), and found that there aren't any of crosscap 2.
- This gives us that there are no local rings of crosscap 2.

Nonlocal rings

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- If we have a product of two rings, $R \times S$, then the product of any two elements of the form $(a, 0)$ and $(0, b)$, where a ranges over all nonzero elements of R and b ranges over all nonzero elements of S, will be zero. Thus we at least have a $K_{|R|-1,|S|-1}$ as a subgraph of the zero-divisor graph of $R \times S$. If R and S are fields, then these will be the only ways to get zero, so the zero-divisor graph will be precisely $K_{|R|-1,|S|-1}$. In this case we get crosscap 2 only in the cases of $\mathbb{F}_4 \times \mathbb{F}_7$ (with a $K_{3,6}$) and $\mathbb{F}_5 \times \mathbb{F}_5$ (with a $K_{4,4}$).

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- If R and S aren't fields, then $(a, b) \cdot (c, d) = 0 \iff ac = 0$ and $bd = 0$. In this case the zero-divisor graph is more complicated, and we've found several cases which have crosscap 2.

Euler's formula

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- Euler's formula tells us that for any 2-cell embedding of a graph in an orientable surface of genus g, $V - E + F = 2 - 2g$, where V is the number of vertices in G, E is the number of edges, and F is the number of faces, and for any 2-cell embedding of a graph in a non-orientable surface of crosscap k, $V - E + F = 2 - k$.

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- Manipulating these formulas gives us that $g = \frac{E-V-F+2}{2}$ for any 2-cell embedding of a graph in an orientable surface and $k = E - V - F + 2$ for any 2-cell embedding of a graph in a non-orientable surface.

Bounds on genus

- For any given graph, it is easy to count the number of vertices and edges it has, but we don't know the number of faces an embedding will have until we find the embedding. However, we know that there has to be at least one face (the outside of the graph is considered a face), and we know that each face must have at least three edges (when there are at least three edges), and each edge can only be in the boundary of two faces. Assuming $E \geq 3$, this gives us the bounds $1 \leq F \leq \frac{2}{3}E$.

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- Plugging this into Euler's formula we get that

$$\frac{\frac{1}{3}E - V + 2}{2} \leq g \leq \frac{E - V + 1}{2}$$

for any 2-cell embedding of a graph in an orientable surface and

$$\frac{1}{3}E - V + 2 \leq k \leq E - V + 1$$

for any 2-cell embedding of a graph in a non-orientable surface.

Maximal and minimal genus of a local ring of fixed order

The question has been asked, given a fixed prime power p^n, will a local ring of that order with a zero-divisor graph of the greatest possible genus have a strictly greater genus than a local ring, which isn't a field, with a zero-divisor graph of the least possible genus, or is it possible that all local rings of order p^n have the same genus.

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- Note that if $n = 1$, then p^n is prime, and all rings of prime order are fields, and have empty, and therefore planar, zero-divisor graphs.

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- Note that if $n = 1$, then p^n is prime, and all rings of prime order are fields, and have empty, and therefore planar, zero-divisor graphs.
- If $n = 2$ and if R is a local ring, not a field, of order p^2, then the maximal ideal will have order p and the square of the maximal ideal will be zero, so the zero-divisor graph will be K_{p-1}.

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- Note that if $n = 1$, then p^n is prime, and all rings of prime order are fields, and have empty, and therefore planar, zero-divisor graphs.
- If $n = 2$ and if R is a local ring, not a field, of order p^2, then the maximal ideal will have order p and the square of the maximal ideal will be zero, so the zero-divisor graph will be K_{p-1}.
- So if $n = 1$ or $n = 2$, all rings of order p^n will have the same zero-divisor graph, and in particular their zero-divisor graphs will have the same genus.

Maximal genus of a local ring of fixed order

If we have any ring of order p^n, the order of the maximal ideal will be p^{n-1}, and thus the ring will have exactly p^{n-1} zero-divisors and the zero-divisor graph will have $p^{n-1} - 1$ vertices. Thus the zero-divisor graph is a subgraph of $K_{p^{n-1}-1}$. If we consider the ring $R = {}^{\mathbb{Z}_p[X_1,\ldots,X_{n-1}]}\big/_{(X_1,\ldots,X_{n-1})^2}$, then we have $|R| = p^n$ and $\Gamma(R) = K_{p^{n-1}-1}$ and $\gamma(\Gamma(R)) = \left\lceil \frac{(p^{n-1}-4)(p^{n-1}-5)}{12} \right\rceil = \left\lceil \frac{p^{2n-2}-9p^{n-1}+20}{12} \right\rceil$, and this is the maximal genus of a ring of order p^n.

$\mathbb{Z}_{p^n}$

We can use $\mathbb{Z}_{p^n}$ to obtain an upper bound on the minimal genus of the zero-divisor graph of a ring of order p^n. We can compute the number of edges in the zero-divisor graph of $\mathbb{Z}_{p^n}$ by

$$E = \binom{p^{\lfloor \frac{n}{2} \rfloor} - 1}{2} + \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} (p^{n-i} - p^{n-i-1})(p^i - 1)$$

which simplifies to

$$E = \frac{n-1}{2} p^n - \frac{n}{2} p^{n-1} - \frac{1}{2} p^{\lfloor \frac{n}{2} \rfloor} + 1$$

and since we know that there will be $p^{n-1} - 1$ vertices, plugging this into the inequality we obtain from Euler's formula we get that

$$g \leq \left\lfloor \frac{n-1}{4} p^n - \frac{n+2}{4} p^{n-1} - \frac{1}{4} p^{\lfloor \frac{n}{2} \rfloor} + \frac{3}{2} \right\rfloor$$

where g is the genus of any orientable surface in which $\Gamma(\mathbb{Z}_{p^n})$ has an embedding.

Maximal and minimal genus

So if we let g be the minimal genus of any ring of order p^n, not a field, and G be the maximal genus of any ring of order p^n we get the bound

$$g \leq \left\lfloor \frac{n-1}{4} p^n - \frac{n+2}{4} p^{n-1} - \frac{1}{4} p^{\lfloor \frac{n}{2} \rfloor} + \frac{3}{2} \right\rfloor$$

while

$$G = \left\lceil \frac{p^{2n-2} - 9p^{n-1} + 20}{12} \right\rceil$$

and we have that whenever $n \geq 3$, the bound for g is strictly less than G unless p^n is 8 or 27. When p^n is 8, we do have that K_4 is planar, so $g = G$. However, when p^n is 27, we have that the genus of K_8 is 2 while $\Gamma(\mathbb{Z}_{27})$ is planar, so $g < G$, and when p^n is not 8 or 27, and $n > 2$, the bounds we've established give us that $g < G$. The case of nonorientable embeddings is similar, and the result is analogous.