Strictly Positive Definite Functions on the Circle

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Missouri State REU 2012

Introduction

Definition

A continuous function $f : [0, \pi] \to \mathbb{R}$ is said to be positive definite on S^1 if, for every $N \in \mathbb{N}$ and every set of N points $x_1, \ldots, x_N$ on S^1, the $N \times N$ matrix A with ij entry $A_{ij} = \left(f(d(x_i, x_j))\right)$ is nonnegative definite, i.e.

$$c^T A c = \sum_{i=1}^{N} \sum_{j=1}^{N} c_i c_j f(d(x_i, x_j)) \geq 0, \quad c = (c_1, \ldots, c_N) \in \mathbb{R}^N,$$

where $d(x, y) = \operatorname{Arccos}(x \cdot y)$ is the usual geodesic distance between two points.

Introduction

Definition

We say a function f is strictly positive definite on the circle if the previous inequality is strict and $c_1, \ldots, c_N$ are not all identically zero.

Theorem (Schoenberg)

All positive definite functions on S^m are those of the form

$$f(t) = \sum_{k=0}^{\infty} a_k P_k^{(\lambda)}(cost)$$

where $\lambda = \frac{(m-1)}{2}$, $a_k \geq 0, \sum a_k < \infty$ and $P_k^{(\lambda)}$ are the standard Gegenbauer polynomials normalized so that $P_k^{(\lambda)}(1) = 1$.

Introduction

Definition

Let $\mathbb{Z}_+$ be the set of nonnegative integers. A subset K of $\mathbb{Z}_+$ is said to induce strict-positive-definiteness (which we shall abbreviate as S.P.D.) on the circle if the function

$$t \mapsto \sum_{k \in K} \frac{P_k^{(\lambda)}(cost)}{2^k}$$

is strictly positive definite on S^1.

Theorem (Chen, Menegatto, and Sun)

It is both necessary and sufficient that K contain infinitely many odd and infinitely many even integers to induce S.P.D. on S^m for $m \geq 2$.

Introduction

Remark

However, for the case of the circle, this no longer holds, as it has been verified that the set of positive integers of the form $4k$ *and* $4k+1$ *does not induce S.P.D. on the circle.*

We wish to find a necessary and sufficient conditions for K *to induce S.P.D. on the circle.*

Equivalent probelm

Theorem

The following problem is equivalent to K inducing S.P.D. on the circle:

For what K does one have the property that for any N, and any N distinct points $0 \leq x_1, \ldots, x_n < 1$, and any $k \in K$ satisfying

$$\sum_{j=1}^{N} c_j e^{2\pi i k x_j} = 0$$

implies the c_j's must be zero.

The Necessary Condition

Definition

A subset K of $\mathbb{Z}$ is said to be ubiquitous modulo (or ubiquitous), if for every positive integer N ($N \geq 2$) and every j ($0 \leq j \leq N-1$), there exists a $k \in K$ such that $k \equiv j \,(mod\ N)$.

Proposition

Let $K \subset \mathbb{Z}$ be ubiquitous. Then given any integer $N\,(N \geq 2)$ and $j\,(0 \leq j \leq N-1)$, there exists infinitely many $k \in K$ such that $k \equiv j\,(mod\ N)$.

Proof.

Fix an $N \in \mathbb{N} \, (N \geq 2)$ and a $(0 \leq j \leq N-1)$. We show that for every natural number p, there is a $k \in K, |k| \geq pN$, such that $k \equiv j \, (mod \, N)$. Since K is ubiquitous, there is a $k \in K$ such that

$$k \equiv pN + j \, (mod \, 3pN).$$

That is to say, there is a $q \in \mathbb{Z}$ such that $k = q \cdot (3pN) + pN + j = (3pq + p)N + j$, thus implying $k \equiv j \, (mod \, N)$. From here, one can easily see that $|k| \geq pN$. The proof is complete upon noting that p is an arbitrary natural number.

$\square$

The Necessary Condition

Lemma

Let N be a natural number, and let $\theta_j := p_j/q_j \, (1 \leq j \leq N)$ be N distinctive rational numbers in [0,1). Let Q $(Q \geq N)$ be a common multiple of $q_j \, (1 \leq j \leq N)$. Let $K \subset \mathbb{Z}$. Assume that for each $l = 0, 1, \ldots, N-1$, there is a $k \in K$, such that $k \equiv l(modQ)$. Then the N functions

$$e^{2\pi ikx_1}, \ldots, e^{2\pi ikx_N}$$

are linearly independent on K over the field of complex numbers.

Proof.

Suppose that $c_1, ..., c_N$ are N complex numbers such that

$$\sum_{j=1}^{N} c_j \, e^{2\pi i k \theta_j} = 0, \qquad \forall k \in K.$$

We show that all the c_j's are zero. For each $l = 0, 1, ..., N-1$, select a $k \in K$, such that $k \equiv l (mod\, Q)$. That is, $k = sQ + l$, where s is an integer. We then have, for each $l = 0, 1, ..., N-1$,

$$\sum_{j=1}^{N} c_j e^{2\pi i k \theta_j} = \sum_{j=1}^{N} c_j e^{2\pi i k (sQ+l) \frac{p_j}{q_j}} = \sum_{j=1}^{N} c_j e^{2\pi i l \frac{p_j}{q_j}} = 0$$

Proof.

The $N \times N$ Vandermonde matrix with jl-entries $e^{2\pi i l \frac{p-j}{q_j}}$ $(1 \leq j \leq N, 0 \leq l \leq N-1)$ and N distinct complex numbers $e^{2\pi i l \frac{p-j}{q_j}}$ $(1 \leq j \leq N)$ in the second row has nonzero determinant. Hence, all the c_j's are zero. □

The Necessary Condition

Theorem

Let $K \subset \mathbb{Z}$. The following two statements are equivalent:

(i) For every natural number N and every set of N distinct rational numbers θ_j ($j = 1, ..., N$) in [0, 1), the N functions as shown earlier are linearly independent on K over the field of complex numbers.

(ii) The subset K of $\mathbb{Z}$ is ubiquitous.

Proof.

From the previous lemma, we have that (ii) $\implies$ (i). To show the converse, suppose K were not ubiquitous. Then there is a $N > 1$ and a nonnegative integer l $(0 \leq l \leq N-1)$ for which there is no $k \in K$ such that $k \equiv l (mod\, N)$. For $j = 1, ..., N, let\ c_j := e^{-2\pi i \frac{lj}{N}}$ and $\theta_j := \frac{j}{N}$. Write each $k \in K$ in the form $k = \mu N + \nu,$, where $\nu \neq l$. From this, we obtain

$$\sum_{j=1}^{N} c_j e^{2\pi i k \theta_j} = \sum_{j=1}^{N} e^{-2\pi i \frac{lj}{N}} \cdot e^{2\pi i (\mu N + \nu) \frac{j}{N}} = \sum_{j=1}^{N} e^{2\pi i (\frac{\nu}{N} - \frac{l}{N}) j}$$

$$= e^{2\pi i (\frac{\nu - l}{N})} \cdot \frac{1 - e^{2\pi i (\frac{\nu - l}{N}) N}}{1 - e^{2\pi i (\frac{\nu - l}{N})}} = 0$$

□

Corollary

Let $K \subset \mathbb{Z}_+$. In order that K induce S.P.D. on the circle, it is necessary that $\overline{K}$ be ubiquitous.

Proof.

Suppose $\overline{K}$ were not ubiquitous. Then by the previous theorem, there exists a N ($N \geq 2$) distinct rational numbers θ_j in [0, 1) such that the N functions given previously are linearly dependent on $\overline{K}$ over the field of complex numbers. By the earlier proposition and theorem, the subset $K \subset \mathbb{Z}_+$ does not induce S.P.D. on the circle.

□

Definition

Let K be a subset of $\mathbb{Z}$. If $\tau_1, ..., \tau_M$ are arbitrary real numbers, if $\theta_1, ..., \theta_M$ are irrational numbers such that the (M + 1) numbers $1, \theta_1, ..., \theta_M$ are linearly independent over the field of rational numbers, and if for every given $\epsilon > 0$, there is an integer $k \in K$ and there are integers $m_1, ..., m_M$ such that

$$|k\theta_j - m - \tau_j| < \epsilon \text{ } for \text{ } j = 1, ..., M,$$

then one says K has the Kronecker approximation property.

Equidistribution

Definition

A sequence of numbers $\xi_1, \xi_2, \ldots, \xi_n, \ldots,$ in [0,1) is said to be equidistributed if for every interval (a,b) $\subset$ [0,1),

$$\lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : \xi_n \in (a, b)\}}{N} = b - a$$

Theorem (Weyl)

If γ is irrational, then the sequence of fractional parts $\langle\gamma\rangle$, $\langle 2\gamma\rangle$, $\langle 3\gamma\rangle$, ... is equidistributed in [0,1).

Theorem (Weyl's criterion)

A sequence of real numbers $\xi_1, \xi_2, \ldots$ in [0,1) is equidistributed if and only if for all integers $k \neq 0$ one has

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0$$

Theorem

Let K be a subset of $\mathbb{Z}_+$. In order that K induce S.P.D. on S^1 it is sufficient that for every $N \geq 2$ and every $j = 1, \ldots, N-1$ the set $\overline{K} \cap \mathbb{Z}_N^j$ has the Kronecker approximation property.

Methods

Remark

We have tried to following methods:

(i) Diophantine Approximations

(ii) Chinese Remainder Theorem

(iii) Complex Analysis

(iv) Almost Periodic Functions

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