Subgroup Graphs of Non-Orientable and Oriented Genus One

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Missouri State REU 2012

Group Theory Review

Definition

A **Group** is a set G along with a binary operation (denoted *) that satisfies the following:

- $\forall a,\ b \in G,\ a * b \in G$
- $\forall a,\ b,\ c \in G$, we have $a * (b * c) = (a * b) * c$
- $\exists$ element $e \in G$ such that $\forall a \in G,\ e * a = a * e = a$
- $\forall a \in G$ there exists a (unique) a^{-1} such that $a * a^{-1} = e$

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Example

The integers are a group under the binary operation of addition. They clearly are closed and associative under addition, and $\forall a \in \mathbb{Z}$, $0 + a = a + 0 = a$. Finally, if $a \in \mathbb{Z}$, then $(-a) + a = 0 = e$, so $(-a)$ is the inverse of a.

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- The even integers are an abelian group under addition - the odd integers are not even a group.
- Any cyclic group; that is, a group generated by one element, is also abelian. $\mathbb{Z}_n$ is an example of a cyclic group, generated by the single element $\overline{1}$.

The Direct Product

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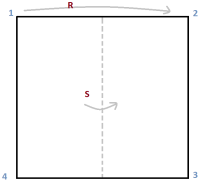
Example

The group $\mathbb{Z}_4 \times \mathbb{Z}_2$ is an abelian group consisting of eight elements: $\{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)\}$.

Dihedral Group: D_8

An example of a group which is non-abelian is the dihedral group with 8 elements, D_8:



D_8 is the group of symmetries of a square. We denote the 90° rotation as r and the horizontal reflection as s, writing the group as follows:

$$D_8 = \langle r, s | r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle$$

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Definition

A subset H of a group G is said to be a **subgroup** of G (denoted $H < G$) if H is a group under same operation as G.

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A quick way to test whether a subset of a group is in fact a subgroup is the following: H $\subset G$ is a subgroup if it satisfies the following:

1. H is non-empty.
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Example

Given $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$, (i.e., the set of even integers) it can easily be shown $H < G$.

Subgroup of D_8

To provide another example of a subgroup, we consider subsets of D_8. We can show that $\langle r \rangle$, i.e. the set which includes only the powers of rotations $\{1, r, r^2, r^3\}$ is a subgroup of D_8:

Proof.

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2. Now we must show, $\forall a, b \in \langle r \rangle$ we have $a * b^{-1} \in \langle r \rangle$. So let r^i, r^j be in $\langle r \rangle$.
3. Then $(r^j)^{-1} = r^{4-j}$, and so we have $r^i (r^j)^{-1} = r^{i+4-j} \in \langle r \rangle$.

□

Definition

The **subgroup lattice** of a group G, denoted $\Gamma(G)$, is the graph whose vertices are the subgroups of G, with two subgroups being connected by an edge if and only if one is contained in the other with no intermediate subgroups. That is to say, if N, H are subgroups of G, then $N - H$ is an edge in $\Gamma(G) \Leftrightarrow H \subset N$ and there is no other $K < G$ such that $H \subset K \subset N$.

We include examples of $\Gamma(\mathbb{Z}_6)$ and $\Gamma(D_8)$ below.

Subgroup Lattice Examples

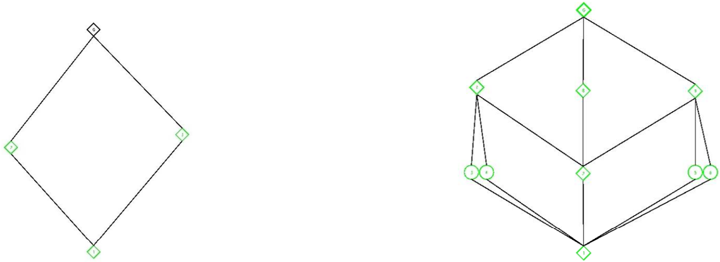

Figure : The subgroup lattices of $\mathbb{Z}_6$ (left), and D_8 (right).

Our Predecessors

In their paper *Finite Groups with Planar Subgroup Lattices*, Bohanon and Reid classified all the finite groups whose subgroup lattices were planar; i.e., could be drawn on the plane with no edges crossing. Our goal is to classify all finite groups whose subgroup lattices have genus one, both oriented and non-orientable; that is to say, those subgroup lattices which can be embedded in the torus and the projective plane, respectively.

Genus Definition

Definition

The **(oriented) genus** of a graph G, which we denote $\gamma(G)$ is the lowest number of handles that must be added to the surface of a sphere in order that the graph be embeddable in that surface.

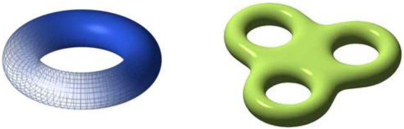

Figure : A torus (Left) with one handle and a 3-torus (Right) with three handles.

The torus and its 2-D representation

We here provide a way of visualizing the representation of the torus in two dimensions:

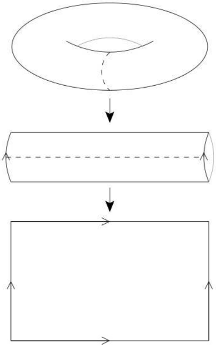

Figure : We thus identify both edges of the two-dimensional surface, first the top and bottom, and then the two sides

Methods of Graph Elimination

We wish to rule out graphs and show that they must have genus greater than one. What tools do we have at our disposal?

1. Lower Bound
2. Obstructions
3. Subgroup graphs and the Lattice Isomorphism Theorem, which we state below:

Theorem (Lattice Isomorphism Theorem)

If H is a normal subgroup of G, then there is a copy of $\Gamma(G/H)$ contained in $\Gamma(G)$.

Genus Lowerbound

Theorem (Euler)

If V denotes the number of vertices in the polyhedron G, E the number of edges, and F the number of faces, then $V - E + F = 2 - 2\gamma(G)$.

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If V denotes the number of vertices in the polyhedron G, E the number of edges, and F the number of faces, then $V - E + F = 2 - 2\gamma(G)$.

We can use this theorem to derive the following lower bound on the genus of a subgroup graph:

Corollary

$$\gamma(G) \geq \lceil 1 + \frac{E}{4} - \frac{V}{2} \rceil.$$

We include the proof immediately following.

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Proof.

- Denote the number of $n - \textit{sided}$ faces by F_n. Since we are dealing with embeddings of subgroup graphs, there are no triangles. So $F = \sum_{i=1}^{n} F_i = F_4 + F_5 + \cdots$.

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- Therefore $F \leq \frac{E}{2}$

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- Note $\sum_{i=1}^{n} iF_i = 2E$
- Thus $2E \geq 4F_4 + 4F_5 + \cdots = 4F$
- Therefore $F \leq \frac{E}{2}$
- Thus $2 - 2\gamma(G) \leq V - E + \frac{E}{2} = V - \frac{E}{2}$, which simplifies to $\gamma(G) \geq 1 + \frac{E}{4} - \frac{V}{2}$.

□

Example of lower bound

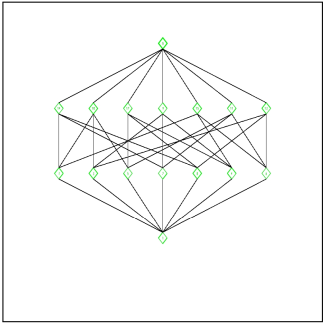

Figure : $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

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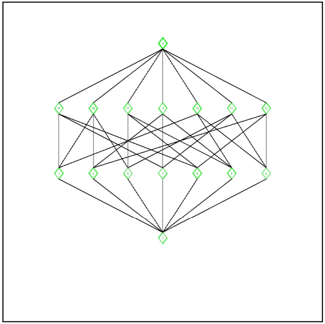

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Applying the lower bound

- $V = 16$

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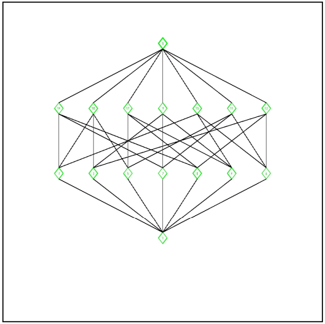

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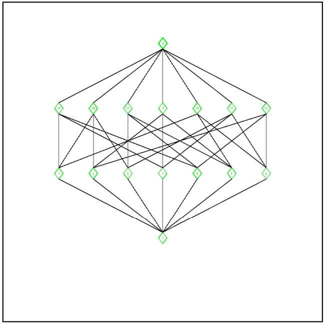

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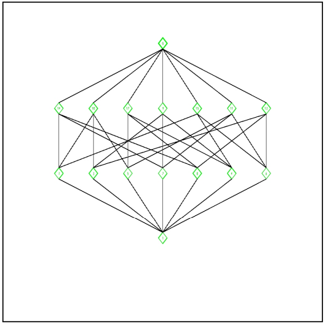

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- $V = 16$
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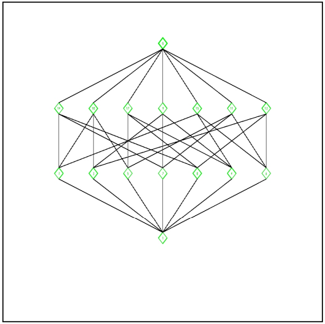
Example of lower bound



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Applying the lower bound

- $V = 16$
- $E = 35$
- $\gamma(G) \geq \lceil 1 + \frac{E}{4} - \frac{V}{2} \rceil$
- $\gamma(G) \geq \lceil 1 + \frac{35}{4} - \frac{16}{2} \rceil$
- $\Rightarrow \gamma(G) \geq 2$

Obstructions for Genus Zero

Another way to limit the number of groups that could have a certain genus is to look for "obstructions" - graphs which cannot be embedded in a given surface. For example, in Bohanon and Reid's paper they utilized the following theorem, which completely describes the obstructions to planarity (genus zero):

Theorem (Kuratowski's Theorem)

A subgroup graph is planar iff it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

We show both of these graphs below.

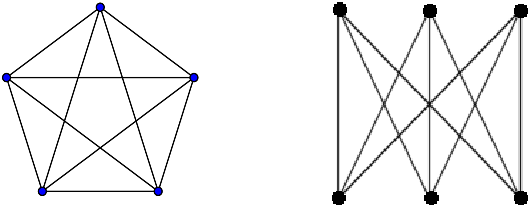


Figure : The complete graph K_5 (left), and the bipartite graph $K_{3,3}$ (right)

Obstructions for Genus One

Obstructions

Among the obstructions which cannot be embedded on the torus are the following:

- $K_{5,4}$
- $K_{3,7}$
- Two copies of $K_{3,3}$

However, unlike in the genus zero case, there are over a hundred thousand other obstructions (up to homeomorphism) to toroidality.

Cyclic Groups

Theorem

A cyclic group G has (oriented) genus one if and only if G is isomorphic to one of the following:

- $\mathbb{Z}_{p^2q^2r}$
- $\mathbb{Z}_{p^3q^2r}$
- $\mathbb{Z}_{p^3q^3r}$
- $\mathbb{Z}_{pqrs}$

where p, q, r, and s are arbitrary primes.

Cyclic Groups Proof

Proof.

By Bohanon we have that $\mathbb{Z}_{p^m}$, $\mathbb{Z}_{p^m q^n}$, $\mathbb{Z}_{p^m qr}$ are planar. We next consider cyclic groups of form $\mathbb{Z}_{p^m q^n r^\ell}$:

- $\mathbb{Z}_{p^2 q^2 r^2}$ contains two copies of $K_{3,3}$, as we show below. Thus we only need to consider cases where $\ell = 1$.
- $\mathbb{Z}_{p^m q^n r}$. If $2 \leq m,n \leq 3$ then $\gamma(Z_{p^m q^n r}) = 1$. We show this by presenting a drawing of $Z_{p^3 q^3 r}$ on the board. Otherwise, $\gamma(\mathbb{Z}_{p^m q^n r}) \geq 2$, as can be seen from the two copies of $K_{3,3}$ found in $\Gamma(\mathbb{Z}_{p^4 q^2 r})$.

Finally, consider $\mathbb{Z}_{p^m q^n r^\ell s^t}$. If $m = n = \ell = t = 1$, then $\gamma(\mathbb{Z}_{p^m q^n r^\ell s^t}) = 1$. Otherwise, the lower bound rules it out. □

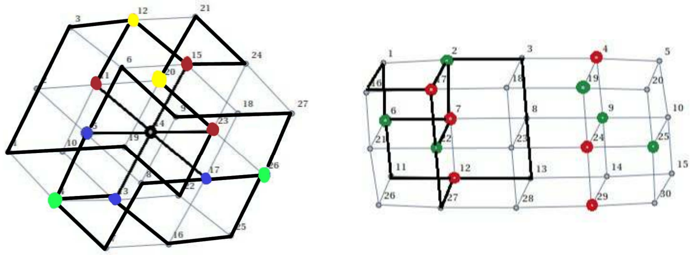
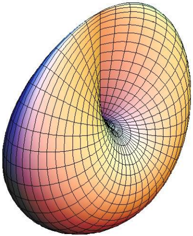


Figure : A pair of $K_{3,3}$'s in the graph of $\mathbb{Z}_{p^2q^2r^2}$ (Left) and in the graph of $\mathbb{Z}_{p^4q^2r}$ (Right).

Definition

The **non-orientable genus** of a graph is the lowest number of cross-caps that must be added to the sphere in order that the graph be embeddable.



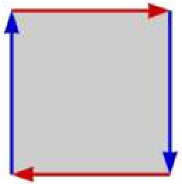


Figure : The projective plane, with one cross-cap, and its 2-D representation

Non-orientable genus example

Example

A graph of non-orientable genus one would be one which can be drawn on the projective plane but not on the sphere. One example is $\mathbb{Z}_4 \times \mathbb{Z}_4$:

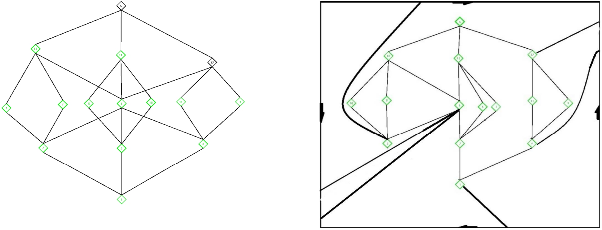


Figure : The regular subgroup lattice of $\mathbb{Z}_4 \times \mathbb{Z}_4$ (left) and drawn on the projective plane (right)

Non-orientable lower bound

For the non-orientable genus we have a lower bound analogous to what we have for the oriented genus. Note that $\tilde{\gamma}(G)$ denotes the non-orientable genus of G.

Theorem (Euler, version two)

If V denotes the number of vertices in a polyhedron G, E the number of edges, and F the number of faces, then $V - E + F = 2 - \tilde{\gamma}(G)$

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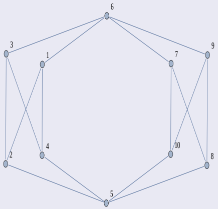
Corollary

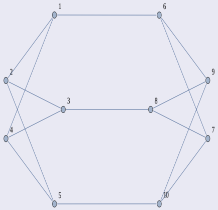
$$\tilde{\gamma}(G) \geq \lceil 2 + \frac{E}{2} - V \rceil$$

Non-Orientable Obstructions

Obstructions

For the projective plane there 35 minor-minimal obstructions. Two such obstructions are the F_6 and G_1 graphs:





Example of F_6 obstruction

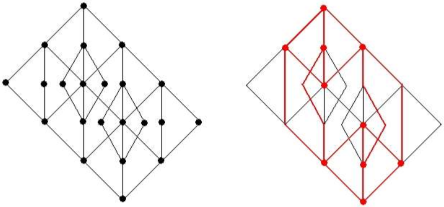


Figure : Graph of $\mathbb{Z}_8 \times \mathbb{Z}_4$ with a F_6 type obstruction

Non-orientable genus of cyclic groups

Theorem

There does not exist any cyclic group of non-orientable genus one.

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Proof.

By Bohanon, all cyclic groups whose orders have fewer than three prime powers are planar, so we need not consider them.
$\mathbb{Z}_{p^\alpha qr}$ is planar. Then we consider $\mathbb{Z}_{p^2q^2r}$, which has a G_1 obstruction and thus has $\tilde{\gamma} \geq 2$. Then since every larger cyclic group whose order has this form must contain $\mathbb{Z}_{p^2q^2r}$ as a subgroup, we are finished with cyclic groups whose order has three prime factors.
Our lower bound rules out cyclic groups whose orders have four or more prime factors, so we are done. □

Abelian Groups of non-orientable genus one

Theorem

Up to isomorphism, the only abelian groups of non-orientable genus one are

- $\mathbb{Z}_4 \times \mathbb{Z}_4$
- $\mathbb{Z}_9 \times \mathbb{Z}_9$

Proof.

- First check abelian groups of prime power order. Bohanon and Reid rule out all of these besides $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. For $p \geq 5$ we have a $K_{3,p+1}$ contained in $\Gamma(\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m})$, so only consider $p = 2$ or $p = 3$. From this we find that $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_9 \times \mathbb{Z}_9$ have non-orientable genus one. We find a G_1 inside $\Gamma(\mathbb{Z}_8 \times \mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_{27} \times \mathbb{Z}_9)$ so we are done with $p = 2$ and $p = 3$. Any further abelian p-group contains a $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, which is ruled out by the lower bound.
- Abelian groups of order $p^m q^n$ are all planar by Bohanon and Reid.
- Abelian groups of order $p^\ell q^m r^n$ are planar if $m,n = 1$, otherwise they contain a G_1. Our lower bound rules out any G with order having four or more distinct prime factors, so we are done.

□

Prime Powers 2^n

Theorem

Let G be a non-abelian group of order 2^n. Then G has non-orientable genus one $\Leftrightarrow$ G is isomorphic to one of the following groups:

- $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b \mid a^4 = b^4 = 1, aba^{-1} = b^3 \rangle$
- $D_{16} = \langle r, s \mid r^8 = s^2 = 1, srs^{-1} = r^{-1} \rangle$
- $Q_{32} = \langle a, b, c \mid a^8 = b^2 = c^2 = abc \rangle$
- $\mathbb{Z}_4 \cdot D_8 = \langle a, b \mid a^8 = 1, a^4 = b^4, bab^{-1} = a^{-1} \rangle$

Prime Powers 3^n

Propositon

Let $|G| = 3^n$ and G non-abelian. Then $\tilde{\gamma}(G) = 1 \Leftrightarrow$ G is isomorphic to one of the following groups:

- $\mathbb{Z}_9 \rtimes \mathbb{Z}_9 = \langle a, b \mid a^9 = 1,\ b^9 = 1, bab^{-1} = a^4 \rangle$
- $(\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3) = \langle a, b, c \mid a^9 = b^3 = 1, ba = ab, c^3 = a^3, cac^{-1} = ab, cbc^{-1} = a^6 b \rangle$
- $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$

Proof for powers of 3

Outline of Proof

Let $n \geq 3$. Then there exists a central subgroup $H < G$ with $H = \langle c \rangle$ and $|H| = 3$, so G/H, has order 3^{n-1}. By the Lattice Isomorphism Theorem, $\tilde{\gamma}(G/H) = 0$ or 1.

1. First, assume $\tilde{\gamma}(G/H) = 0$. Then by Bohanon and Reid G/H must be isomorphic to:
   - $G/H \cong \mathbb{Z}_{3^{n-2}} \times \mathbb{Z}_3 = \langle a, b \mid a^{3^n-2} = 1,\ b^3 = 1, aba^{-1}b^{-1} = 1 \rangle$
   - $G/H \cong M_{3^{n-1}} = \langle a, b \mid a^{3^{n-2}} = b^3 = 1, bab^{-1} = a^{3^{n-3}+1} \rangle$
2. Second, assume $\tilde{\gamma}(G/H) = 1$. Then by induction G/H must be isomorphic to:
   - $\mathbb{Z}_9 \times \mathbb{Z}_9$
   - $\mathbb{Z}_9 \rtimes \mathbb{Z}_9$
   - $(\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3)$
   - $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$

Example ($\tilde{\gamma}(G/H) = 0$)

Suppose $G/H \cong \mathbb{Z}_{3^{n-2}} \times \mathbb{Z}_3$. Then we have a, $b \in G$ such that $a^{3^n-2} = c^i$, $b^3 = c^j$ and $aba^{-1}b^{-1} = c^k$, where $i, j, k \in \{0,1,2\}$. We can dismiss the case where $i \neq 0$, and consider only zero or non-zero values for j and k to get the following possibilities for G:

- $j = k = 0$. Then $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{3^{n-2}} \times \mathbb{Z}_3 \times \mathbb{Z}_3$
- $j \neq 0$, $k = 0$. Then $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{3^{n-2}} \times \mathbb{Z}_{3^2}$. Note that these are planar for $n < 4$, genus one for $n = 4$, and genus ≥ 2 for $n > 4$.
- $j = 0$, $k \neq 0$. Then $G = \langle b, c \rangle \rtimes \langle a \rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_{3^{n-2}}$.
- $j \neq 0$, $k \neq 0$. Then $G = \langle b \rangle \rtimes \langle a \rangle \cong \mathbb{Z}_{3^2} \rtimes \mathbb{Z}_{3^{n-2}}$.

Example ($\tilde{\gamma}(G/H) = 1$)

Consider the first case, where $G/H \cong \mathbb{Z}_9 \times \mathbb{Z}_9$. Then we have $a, b, c \in G$ with $a^9 = c^j$, $b^9 = c^\ell$, $aba^{-1}b^{-1} = c^k$. Assign values for j, k, ℓ as follows:

- $j = k = \ell = 0$. Then $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_3$
- $j = 0,\ k = 0,\ \ell \neq 0$. $G = \langle b, c \rangle \rtimes \langle a \rangle \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_9$
- $j \neq 0,\ k \neq 0,\ \ell = 0$. $G = \mathbb{Z}_{27} \times \mathbb{Z}_9$.
- $j = 0,\ k \neq 0,\ \ell \neq 0$. $G = \langle b \rangle \rtimes \langle a \rangle \cong \mathbb{Z}_{27} \rtimes \mathbb{Z}_9$. G is the same (switching a and b) if $j \neq 0$ and $k = 0$.
- $j = 0,\ k \neq 0,\ \ell = 0$. $G = \langle b \rangle \times \langle a \rangle \cong \mathbb{Z}_{27} \times \mathbb{Z}_9$. Again G is the same (switching a and b) if $j \neq 0$ and $k = 0$.
- $j \neq 0,\ k \neq 0,\ \ell \neq 0$. $G = \langle a \rangle \rtimes \langle ba^{-1} \rangle \cong \mathbb{Z}_{27} \rtimes \mathbb{Z}_9$

Powers of primes greater than 3

Proposition

Let G be a non-abelian group, and $|G| = p^n$ for a prime $p \geq 5$ Then $\tilde{\gamma}(G) \neq 1$.

Proof for powers of primes greater than 3

Proof.

1. For $n = 1$ or 2, $|G| = p^n \Rightarrow G$ abelian.
2. For $n = 3$ there are five groups of order p^n: $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, the modular group M_{p^3}, and the group of 3×3 matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ with a,b,c in $\mathbb{Z}_p$. The first two and the fourth are planar, the third is ruled out by the lower bound. The fifth contains a $K_{3,p}$ where G, $Z(G)$, and $\{e\}$ make up the three elements of degree p and the p copies of $\mathbb{Z}_p \times \mathbb{Z}_p$ make up the p elements of order 3. So for $p \geq 5$ we have $\tilde{\gamma}(G) \geq 2$.
3. For $n \geq 4$ it is easy to extend the proof given in Bohanon-Reid, so we are done.

Groups of two prime power orders

Having finished with groups of order p^n, we proceed to groups whose orders are made up of powers of two different primes.

Theorem

Let G be a non-abelian group with $|G| = p^\alpha q^\beta$. Then $\tilde{\gamma}(G) = 1$ if and only if G is isomorphic to one of the following:

- *$SL(2,3)$, the special linear group of 2×2 matrices with values in $\mathbb{F}_3$.*
- *$\mathbb{Z}_9 \rtimes \mathbb{Z}_{2^n}$, for arbitrary $n \in \mathbb{N}$.*

Note that (much to our surprise), there is an infinite family of groups with non-orientable genus one; specifically the second item mentioned in the theorem. This family includes D_{18} - the subgroup lattices of each member of this family includes a copy of $\Gamma(D_{18})$ and a number of planar spindles.

Examples of our infinite family

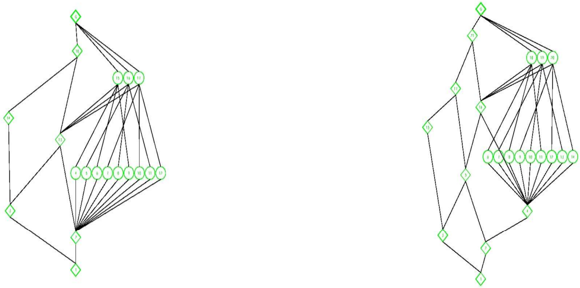

Figure : A copy of $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$ (left), and $\mathbb{Z}_9 \rtimes \mathbb{Z}_8$ (right)

Groups of three prime power orders

Conjecture

We conjecture there are no non-orientable genus one subgroup graphs of order $p^\alpha q^\beta r^\gamma$

Groups of four prime power orders

Theorem

If G is solvable of and the order of G has four or more prime powers, then $\tilde{\gamma}(G) > 1$.

Proof.

In their paper, Bohanon and Reid show that G will have a Sylow basis consisting of four sylow subgroups. Then the sublattice of $\Gamma(G)$ composed of those four groups and their products makes up a graph homeomorphic to $\Gamma(\mathbb{Z}_{pqrs})$, which has non-orientable genus greater than one. □

The Next Frontier

What's Next?

So what are the next steps on this road of inquiry? We'd like to prove our conjecture for groups whose orders have three prime powers, wrap up our proof for groups whose orders have two prime powers, and condense some of the cases for our proofs of prime powers.

Beyond this, we can look to continue investigating the oriented genus of subgroup graphs, a topic which we've already resolved if the group is cyclic. However, the question still lies open for non-cyclic abelian groups and non-abelian groups.

Some References

[1] W. Tutte, *Graph Theory*. Cambridge University Press: 29 January, 2001.

[2] Joseph Bohanon and Les Reid, *Finite Groups with Planar Subgroup Lattices.* J Algebr Comb, **23** (2006) 207-223

[3] W. Burnside, *Theory of Groups of Finite Order*. Dover Publications: 1955.