A Topic on Constructive Theory of Functions

Most calculus textbooks include a theorem stating that differentiable functions are continuous. To show that the converse is not true, the example $f(x) = |x|$ is often used. This function is continuous everywhere, but not differentiable at $x = 0$. It is a simple task to construct continuous functions that are not differentiable at each point of a countable set. However, most mathematicians of the nineteenth century had the impression that continuous functions are “mostly” differentiable [1]. In the early 1870’s, Weierstrass [3] startled the mathematical world by constructing a function that is everywhere continuous but nowhere differentiable. Other examples soon followed. Furthermore, using a standard category argument, one can show that continuous functions that are differentiable somewhere belong to a “tiny minority”.

In order for a function $f(x)$ to be continuous at a point, it is necessary (but not sufficient) that the limit of $f(x)$ exists at the point. One naturally asks the following question: Are there any functions that have a limit everywhere but are continuous nowhere? In her master’s thesis under the guidance of the author, Julie Millett [2] gave a negative answer to the above question. In fact, she proved the following surprising result: Let $A$ be a dense subset of the closed interval $[a,b]$. Assume that $f(x)$ has a limit at each point of $A$. Then there exists a dense subset $B$ of $[a,b]$ such that $f(x)$ is continuous at every point of $B$. Furthermore, the set $B$ has the same cardinality as that of the set of all real numbers.

The author proposes to generalize Millett’s result to functions defined on compact subsets of certain types of topological spaces. On the one hand, Millett’s result does not always hold true on a compact set of a general Hausdorff space. On the other hand, it is true on a compact set of a completely normal topological space. Students can investigate several interesting topics using various strength of the countability and the separation axioms of topological spaces. These research topics are suitable for students who are interested in analysis, topology and foundation of mathematics. Students who finish a project will have a solid grasp of the basic constructive theory of functions and a thorough understanding of the countability and the separation axioms in topology.

Prerequisites: A basic background in calculus, and optimally, some experience with real analysis and topological spaces.

References