Characteristic Polynomials

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Adjacency Matrix

Definition

Adjacency Matrix:
Let Γ be a graph with n vertices a_i. An adjacency matrix A for Γ is $n \times n$ such that

$$A_{ij} = 1 \text{ if there is an edge between } a_i \text{ and } a_j$$

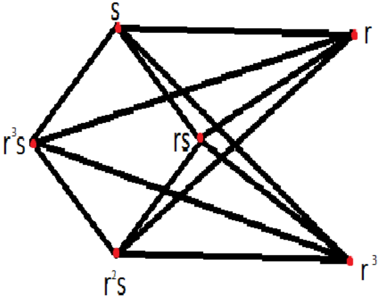
and

$$A_{ij} = 0 \text{ if there is no edge between } a_i \text{ and } a_j.$$

To find the adjacency matrix for paths of lengths k between a_i and a_j, take the kth power of A.

$\Gamma(D_8)$ Adjacency Matrix

Non-Commuting $\Gamma(D_8) \cong K_{2,2,2}$



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Characteristic Polynomial

Definition

Characteristic Polynomial:
The characteristic polynomial of the matrix A is

$$det(A - \lambda I).$$

Complete K-Partite Graph:
A graph is complete K-partite denoted by $K(n_1, ..., n_k)$ if the n_i vertices in the ith class are not connected but all n_i vertices are connected to the remaining $n - n_i$ vertices. If there are j parts of size n_1 for a complete K-partite graph, the graph can be represented by $K(n_1^j, ..., n_k)$.

Proving Characteristic Polynomials

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- Prove conjectured polynomials by inducting upon the expansions found.

Characteristic Polynomial Findings

Characteristic Polynomial of Complete K-Partite:

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Characteristic Polynomial of Complete K-Partite:

- $K_{m_1,m_2,\ldots,m_k} : (-1)^{m_1+m_2+\ldots+m_k}\ \lambda^{m_1+m_2+\ldots+m_k-k}(\lambda^k - \sum_{i=2}^{i=k} (i-1)\ \sigma_i(m_1, m_2, \ldots, m_k)\ \lambda^{k-i})$

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  - $\sigma_i(m_1,m_2,\dots,m_k)$ are called symmetric polynomials, which follow the form

$$\sigma_1(m_1,m_2,\dots,m_k) = m_1 + m_2 + \dots + m_k$$
$$\sigma_2(m_1,m_2,\dots,m_k) = m_1m_2 + m_1m_3 + \dots + m_{k-1}m_k$$
$$\dots$$
$$\sigma_k(m_1,m_2,\dots,m_k) = m_1m_2\dots m_k.$$

Graph, Adjacency Matrix, and Characteristic Polynomial of $\Gamma(K_{2,3,4})$

$$\sigma_1(2,3,4) = 2+3+4 = 9$$

$$\sigma_2(2,3,4) = 2*3+2*4+3*4 = 26$$

$$\sigma_3(2,3,4) = 2*3*4 = 24$$

$$\begin{aligned} p(\lambda) =& (-1)^9\lambda^6(\lambda^3 - \sum_{i=2}^{i=3}(i-1)\sigma_i(2,3,4)\lambda^{3-i}) \\ =& -\lambda^6(\lambda^3 - (6\lambda + 8\lambda + 12\lambda) - 2(24)) \\ =& -\lambda^6(\lambda^3 - 26\lambda - 48) \end{aligned}$$

Eigenvalues

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The eigenvalues of a matrix A are the solutions to the characteristic polynomial $p(\lambda) = 0$.

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- If H is an induced subgraph of Γ, then $\mu_{min}(\Gamma) \leq \mu_{min}(H) \leq \mu_{max}(H) \leq \mu_{max}(\Gamma)$.

AC Group

Definition

AC Group:
A group G is an AC Group provided that

$$C_G(g) = \{x | xg = gx\}$$

is an abelian subgroup of G, $\forall g \in G \setminus Z(G)$.

Graphs of AC Groups

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- The non-commuting graphs of AC groups are complete K-partite graphs. In fact, the non-commuting graph of a group is complete K-partite if and only if the group is an AC group.
- Through this fact, once the corresponding K-partite graph has been determined for a specific AC group, we know its characteristic polynomial.

K-Partite Graphs with Groups

Determining which graphs $K_{m_1 z, m_2 z, \ldots, m_k z}$ where $z = |Z(G)|$ and $k = |G/Z(G)|$ can be graphs of AC groups using the following:

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- The corresponding group G must be capable.
  - A group G is capable if there exists a group H such that $G \cong \frac{H}{Z(H)}$.

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Which groups can be eliminated?

- Cyclic groups G are not capable.
- If $|G| = pq$, where p, q are distinct primes with $p < q$, if p does not divide $(q-1)$, then $\Gamma(G) \cong \mathbb{Z}_{pq}$, which is not capable.

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  - For odd n: $K_{1^n,n-1}$
- Quaternion Groups
- Capable Semidirect Products
  - For example: $\mathbb{Z}_7 \rtimes \mathbb{Z}_3, \mathbb{Z}_3 \rtimes Q_8, (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_8$

Results

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- No complete or complete bipartite, proven in [4].
- Identified all K-partite for $k = 3, 4, 5, 6$.
- K-partite for $k = 7, 8, 9, 10$ in progress.

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  - From a group theoretical standpoint, this cannot occur, as G would be a proper subgroup of itself.
- No complete bipartite non-commuting graphs.
  - This is again impossible, though group theoretical reasoning. G would be union of two proper subgroups-centralizers $C_G(x)$ and $C_G(y)$, but elements xy cannot be placed.

Table: K-Partite Graphs for k=3,4,5,6

Graph	Groups, G
K_{z^3}	D_8, Q_8
$K_{z^3,2z}$	D_6
$K_{(2z)^4}$	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$
$K_{z^4,3z}$	D_{16}, QD_{16}, Q_{16}
$K_{(2z)^4,3z}$	A_4
$K_{(3z)^5}$	$((\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2*$
$K_{z^5,4z}$	D_{10}
$K_{(3z)^4,4z}$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$
$K_{(4z)^6}$	$\mathbb{Z}_{25} \rtimes \mathbb{Z}_5, (\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_5$

Semidirect Products

- $\mathbb{Z}_p \rtimes \mathbb{Z}_q = \langle a, b | a^p = b^q = 1, bab^{-1} = a^k \rangle$ where $k^q \equiv 1 \ mod \ p$ with p, q distinctly prime, AC.

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  - p, q, r, s relatively prime, not AC
  - $p = q, r = s$ relatively prime, AC.

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  - For example, $\mathbb{Z}_{35} \rtimes \mathbb{Z}_6 \cong D_{10} \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ is not AC.

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  - For example, $\mathbb{Z}_{35} \rtimes \mathbb{Z}_6 \cong D_{10} \times (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$ is not AC.
- Conjecture: If the semidirect product $\mathbb{Z}_m \rtimes \mathbb{Z}_n$ is AC, then its graph will be of the form $K_{m-|Z(G)|}K_{|Z(G)|(n-1)}...K_{|Z(G)|(n-1)}$ with $\frac{m}{|Z(G)|}$ copies of $K_{|Z(G)|(n-1)}$.

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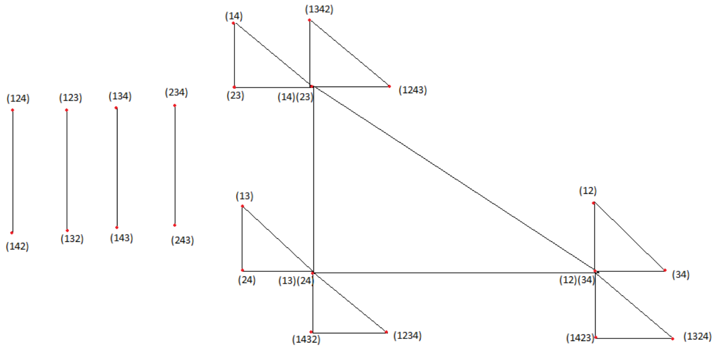
  - $S_4 : \lambda^{11}(\lambda+2)^6(\lambda^2+2\lambda-4)^2(\lambda^3-16\lambda^2-76\lambda-48)$
  - $S_5 : \lambda^{65}(\lambda+4)^{10}(\lambda^2+2\lambda-2)^5(\lambda^3-106\lambda^2-896\lambda-1680)(\lambda^3+4\lambda^2-6\lambda-10)^4(\lambda^4+8\lambda^3+10\lambda^2-28\lambda-40)^5$

$\Gamma(S_4)$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