Eulerian non-commuting graphs

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- A graph is Eulerian if and only if every vertex has even degree and path Eulerian if and only if exactly two vertices have odd degree.
- For an element $v \in G$, the degree of the vertex corresponding to v in $\Gamma(G)$, the non-commuting graph of G, is $d(v) = |G| - |C_G(v)|$.

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- $|[x]| = \frac{|G|}{|C_G(x)|}$

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- If $|G| = 2^n$, then $|C_G(v)| = 2^m$ for some nonnegative integer $m \leq n$. $C_G(v)$ must contain at least e and v, so $m \geq 1$ (when $v = e$, $C_G(v) = G$). Therefore, $|G|$ is even and $|C_G(v)|$ is even for all $v \in G$, so $d(v) = |G| - |C_G(v)|$ is even for all $v \in G$ and $\Gamma(G)$ is Eulerian.

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- If $|G|$ is even and $|H|$ is odd, then $|C_G(v)|$ must be even for all $v \in G$, so $\Gamma(G \times H)$ is Eulerian if and only if $\Gamma(G)$ is Eulerian.
- If $|G|$ and $|H|$ are both even, then it must be true that either $|C_G(v)|$ is even for all $v \in G$ or $|C_H(w)|$ is even for all $w \in H$, so $\Gamma(G \times H)$ is Eulerian if and only if either $\Gamma(G)$ is Eulerian or $\Gamma(H)$ is Eulerian.

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- Therefore, $\Gamma(S_3)$ is path Eulerian and $\Gamma(S_n)$ is non-Eulerian for all larger n.

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- If n is even, then $|C_{D_{2n}}(r^i)| = n$ $(1 \leq i \leq \frac{n}{2} - 1$ or $\frac{n}{2} + 1 \leq i \leq n-1)$ and $|C_{D_{2n}}(r^i s)| = 4$ $(0 \leq i \leq n-1)$, so $\Gamma(D_{2n})$ is Eulerian.

Semidirect products

- $G \cong \mathbb{Z}_m \rtimes \mathbb{Z}_{2^n}$, where $m = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$

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- $\Gamma(G)$ Eulerian if and only if $\Gamma(\mathbb{Z}_{u_1 u_2 \ldots u_n} \rtimes \mathbb{Z}_{2^n})$ Eulerian

Semidirect products (continued)

- $\mathbb{Z}_{u_1 u_2 \ldots u_n} \rtimes \mathbb{Z}_{2^n} = \langle a_1, a_2, \ldots, a_n, b | a_i^{u_i} = b^{2^n} = 1, a_j a_i = a_i a_j, b a_i = a_i^{l_i} b \rangle$, where $o_{u_i}(l_i) = 2^i$

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- For any y, let s be the largest power of 2 that divides y (i.e. 2^s divides y but 2^{s+1} does not). Then, the values of $x_{s+1}', x_{s+2}', \ldots, x_{n-1}'$ are forced.
- If r is the smallest integer less than or equal to s for which $x_{r+1} = x_{r+2} = \ldots = x_s = 0$, then 2^r divides y'.
- Then, $|C_G(a_1^{x_1} a_2^{x_2} \ldots a_{n-1}^{x_{n-1}} b^y)| = 2^{n-r} u_1 u_2 \ldots u_s$. $r \leq n-1$, so $|C_G(a_1^{x_1} a_2^{x_2} \ldots a_{n-1}^{x_{n-1}} b^y)|$ is always even.

Semidirect products (continued)

- $\mathbb{Z}_{u_1 u_2 \ldots u_n} \rtimes \mathbb{Z}_{2^n} = \langle a_1, a_2, \ldots, a_n, b | a_i^{u_i} = b^{2^n} = 1, a_j a_i = a_i a_j, b a_i = a_i^{l_i} b \rangle$, where $o_{u_i}(l_i) = 2^i$
- Suppose $a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n} b^y$ and $a_1^{x_1'} a_2^{x_2'} \ldots a_n^{x_n'} b^{y'}$ commute. Then, $x_i(l_i^{y'} - 1) \equiv x_i'(l_i^y - 1)$ for all $0 \leq i \leq n$.
- If $u_n \neq 1$, then for all elements of G with $y = 0$ and $x_n \neq 0$, $|C_G(a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n})| = u_1 u_2 \ldots u_n$, which is odd.
- If $u_n = 1$, then the congruence for $i = n$ is trivially true.
- For any y, let s be the largest power of 2 that divides y (i.e. 2^s divides y but 2^{s+1} does not). Then, the values of $x'_{s+1}, x'_{s+2}, \ldots, x'_{n-1}$ are forced.
- If r is the smallest integer less than or equal to s for which $x_{r+1} = x_{r+2} = \ldots = x_s = 0$, then 2^r divides y'.
- Then, $|C_G(a_1^{x_1} a_2^{x_2} \ldots a_{n-1}^{x_{n-1}} b^y)| = 2^{n-r} u_1 u_2 \ldots u_s$. $r \leq n - 1$, so $|C_G(a_1^{x_1} a_2^{x_2} \ldots a_{n-1}^{x_{n-1}} b^y)|$ is always even.
- Therefore, $\Gamma(\mathbb{Z}_m \rtimes \mathbb{Z}_{2^n})$ is Eulerian if and only if $o_{p_i}(k_i) \neq 2^n$ for all primes $p_i | m$.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^m r} \rtimes \mathbb{Z}_{2^n}$, where $r = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^m r} \rtimes \mathbb{Z}_{2^n}$, where $r = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^m} = b_i^{p_i^{e_i}} = c^{2^n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, cb_i = b_i^{k_i} c \rangle$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^m r} \rtimes \mathbb{Z}_{2^n}$, where $r = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^m} = b_i^{p_i^{e_i}} = c^{2^n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)$ divides $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)|$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^m r} \rtimes \mathbb{Z}_{2^n}$, where $r = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^m} = b_i^{p_i^{e_i}} = c^{2^n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)$ divides $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)|$
- $(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)^t = a^{u(1+k_0^w+k_0^{2w}+\ldots+k_0^{(t-1)w})} b_1^{v_1(1+k_1^w+k_1^{2w}+\ldots+k_1^{(t-1)w})} \ldots c^{tw}$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^m r} \rtimes \mathbb{Z}_{2^n}$, where $r = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^m} = b_i^{p_i^{e_i}} = c^{2^n} = 1, b_i a = ab_i, b_j b_i = b_i b_j, ca = a^{k_0} c, cb_i = b_i^{k_i} c \rangle$
- $o(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)$ divides $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)|$
- $(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)^t = a^{u(1+k_0^w+k_0^{2w}+\ldots+k_0^{(t-1)w})} b_1^{v_1(1+k_1^w+k_1^{2w}+\ldots+k_1^{(t-1)w})} \ldots c^{tw}$
- $o(c^w)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)$.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^m r} \rtimes \mathbb{Z}_{2^n}$, where $r = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^m} = b_i^{p_i^{e_i}} = c^{2^n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)$ divides $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)|$
- $(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)^t = a^{u(1+k_0^w+k_0^{2w}+\ldots+k_0^{(t-1)w})} b_1^{v_1(1+k_1^w+k_1^{2w}+\ldots+k_1^{(t-1)w})} \ldots c^{tw}$
- $o(c^w)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)$.
- $o(c^w)$ is a power of 2. Therefore, if $w \neq 0$, then $o(c^w)$ is even, so $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots c^w)|$ is even and $\Gamma(G)$ is Eulerian.

Semidirect products (continued)

- If $w = 0$, then $o(a^u)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \ldots)$.

Semidirect products (continued)

- If $w = 0$, then $o(a^u)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \ldots)$.
- $o(a^u)$ is a power of 2. Therefore, if $u \neq 0$, then $o(a^u)$ is even, so $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots)|$ is even and $\Gamma(G)$ is Eulerian.

Semidirect products (continued)

- If $w = 0$, then $o(a^u)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \ldots)$.
- $o(a^u)$ is a power of 2. Therefore, if $u \neq 0$, then $o(a^u)$ is even, so $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots)|$ is even and $\Gamma(G)$ is Eulerian.
- If $u = 0$, then $o(b_1^{v_1} b_2^{v_2} \ldots)$ is odd. However, $C_G(b_1^{v_1} b_2^{v_2} \ldots) \geq \langle a, b_1, b_2, \ldots \rangle$, so $|C_G(b_1^{v_1} b_2^{v_2} \ldots)|$ is even and $\Gamma(G)$ is Eulerian.

Semidirect products (continued)

- If $w = 0$, then $o(a^u)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \dots)$.
- $o(a^u)$ is a power of 2. Therefore, if $u \neq 0$, then $o(a^u)$ is even, so $|C_G(a^u b_1^{v_1} b_2^{v_2} \dots)|$ is even and $\Gamma(G)$ is Eulerian.
- If $u = 0$, then $o(b_1^{v_1} b_2^{v_2} \dots)$ is odd. However, $C_G(b_1^{v_1} b_2^{v_2} \dots) \geq \langle a, b_1, b_2, \dots \rangle$, so $|C_G(b_1^{v_1} b_2^{v_2} \dots)|$ is even and $\Gamma(G)$ is Eulerian.
- Therefore, $\Gamma(G)$ is Eulerian.

Semidirect products (continued)

- If $w = 0$, then $o(a^u)$ divides $o(a^u b_1^{v_1} b_2^{v_2} \ldots)$.
- $o(a^u)$ is a power of 2. Therefore, if $u \neq 0$, then $o(a^u)$ is even, so $|C_G(a^u b_1^{v_1} b_2^{v_2} \ldots)|$ is even and $\Gamma(G)$ is Eulerian.
- If $u = 0$, then $o(b_1^{v_1} b_2^{v_2} \ldots)$ is odd. However, $C_G(b_1^{v_1} b_2^{v_2} \ldots) \geq \langle a, b_1, b_2, \ldots \rangle$, so $|C_G(b_1^{v_1} b_2^{v_2} \ldots)|$ is even and $\Gamma(G)$ is Eulerian.
- Therefore, $\Gamma(G)$ is Eulerian.
- $\Gamma(\mathbb{Z}_{2m} \rtimes \mathbb{Z}_{2^n})$ is Eulerian for all positive integers m and n.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n}$, where $m = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$ and $n = q_1^{f_1} q_2^{f_2} \ldots$ for distinct odd primes $q_1, q_2, \ldots$ and positive integers $f_1, f_2, \ldots$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n}$, where $m = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$ and $n = q_1^{f_1} q_2^{f_2} \ldots$ for distinct odd primes $q_1, q_2, \ldots$ and positive integers $f_1, f_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^r} = b_i^{p_i^{e_i}} = c^{2n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n}$, where $m = p_1^{e_1} p_2^{e_2} \dots$ for distinct odd primes $p_1, p_2, \dots$ and positive integers $e_1, e_2, \dots$ and $n = q_1^{f_1} q_2^{f_2} \dots$ for distinct odd primes $q_1, q_2, \dots$ and positive integers $f_1, f_2, \dots$
- $G \cong \langle a, b_1, b_2, \dots, c | a^{2^r} = b_i^{p_i^{e_i}} = c^{2n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(k_0)$ divides 2^r and $2n$, so $o(k_0) = 1, 2$.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n}$, where $m = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$ and $n = q_1^{f_1} q_2^{f_2} \ldots$ for distinct odd primes $q_1, q_2, \ldots$ and positive integers $f_1, f_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^r} = b_i^{p_i^{e_i}} = c^{2n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(k_0)$ divides 2^r and $2n$, so $o(k_0) = 1, 2$.
- If $o(k_0) = 1$, then $k_0 = 1$, so $\mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n} \cong \mathbb{Z}_{2^r} \times (\mathbb{Z}_m \rtimes \mathbb{Z}_{2n})$ and $\Gamma(G)$ is Eulerian.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n}$, where $m = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$ and $n = q_1^{f_1} q_2^{f_2} \ldots$ for distinct odd primes $q_1, q_2, \ldots$ and positive integers $f_1, f_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^r} = b_i^{p_i^{e_i}} = c^{2n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(k_0)$ divides 2^r and $2n$, so $o(k_0) = 1, 2$.
- If $o(k_0) = 1$, then $k_0 = 1$, so $\mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n} \cong \mathbb{Z}_{2^r} \times (\mathbb{Z}_m \rtimes \mathbb{Z}_{2n})$ and $\Gamma(G)$ is Eulerian.
- Now, let $o(k_0) = 2$.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n}$, where $m = p_1^{e_1} p_2^{e_2} \ldots$ for distinct odd primes $p_1, p_2, \ldots$ and positive integers $e_1, e_2, \ldots$ and $n = q_1^{f_1} q_2^{f_2} \ldots$ for distinct odd primes $q_1, q_2, \ldots$ and positive integers $f_1, f_2, \ldots$
- $G \cong \langle a, b_1, b_2, \ldots, c | a^{2^r} = b_i^{p_i^{e_i}} = c^{2n} = 1, b_i a = a b_i, b_j b_i = b_i b_j, ca = a^{k_0} c, c b_i = b_i^{k_i} c \rangle$
- $o(k_0)$ divides 2^r and $2n$, so $o(k_0) = 1, 2$.
- If $o(k_0) = 1$, then $k_0 = 1$, so $\mathbb{Z}_{2^r m} \rtimes \mathbb{Z}_{2n} \cong \mathbb{Z}_{2^r} \times (\mathbb{Z}_m \rtimes \mathbb{Z}_{2n})$ and $\Gamma(G)$ is Eulerian.
- Now, let $o(k_0) = 2$.
- Define $u_g = \prod_{o_{p_i^{e_i}}(k_i) = g} p_i^{e_i}$ for all $g | 2n$, except

$$u_2 = 2^r \prod_{o_{p_i^{e_i}}(k_i) = 2} p_i^{e_i}.$$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{u_1} \times (\mathbb{Z}_{\frac{2^r m}{u_1}} \rtimes \mathbb{Z}_{2n})$, so if $H = \mathbb{Z}_{\frac{2^r m}{u_1}} \rtimes \mathbb{Z}_{2n}$, then $\Gamma(G)$ is Eulerian if and only if $\Gamma(H)$ is Eulerian.

Semidirect products (continued)

- $G \cong \mathbb{Z}_{u_1} \times (\mathbb{Z}_{\frac{2^r m}{u_1}} \rtimes \mathbb{Z}_{2n})$, so if $H = \mathbb{Z}_{\frac{2^r m}{u_1}} \rtimes \mathbb{Z}_{2n}$, then $\Gamma(G)$ is Eulerian if and only if $\Gamma(H)$ is Eulerian.
- $H \cong \langle a_g, b | a_g^{u_g} = b^{2n} = 1, a_h a_g = a_g a_h, ba_g = a_g^{l_g} b \rangle$, where g takes all values satisfying $g|2n$ except 1 and $o_{u_g}(l_g) = g$

Semidirect products (continued)

- $G \cong \mathbb{Z}_{u_1} \times (\mathbb{Z}_{\frac{2^r m}{u_1}} \rtimes \mathbb{Z}_{2n})$, so if $H = \mathbb{Z}_{\frac{2^r m}{u_1}} \rtimes \mathbb{Z}_{2n}$, then $\Gamma(G)$ is Eulerian if and only if $\Gamma(H)$ is Eulerian.
- $H \cong \langle a_g, b | a_g^{u_g} = b^{2n} = 1, a_h a_g = a_g a_h, b a_g = a_g^{l_g} b \rangle$, where g takes all values satisfying $g|2n$ except 1 and $o_{u_g}(l_g) = g$
- $\gamma = \left(\prod_{i|2n, i \neq 1} a_i^{x_i} \right) b^y$ and $\gamma' = \left(\prod_{i|2n, i \neq 1} a_i^{x_i'} \right) b^{y'}$ commute if and only if $x_g(l_g^{y'} - 1) \equiv x_g'(l_g^y - 1)$ mod u_g for all $g|2n$, $g \neq 1$.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.
- If y' is even, then there are no restrictions on x_2'. There are n ways to choose y' and u_2 ways to choose x_2', so $u_2 n$ divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.
- If y' is even, then there are no restrictions on x_2'. There are n ways to choose y' and u_2 ways to choose x_2', so $u_2 n$ divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2, then all choices of x_2' commute with x_2, so u_2 divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.
- If y' is even, then there are no restrictions on x_2'. There are n ways to choose y' and u_2 ways to choose x_2', so $u_2 n$ divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2, then all choices of x_2' commute with x_2, so u_2 divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If y is odd, then $x_2' \equiv x_2 \left(\frac{l_2^{y'}-1}{\gcd(l_2-1,u_2)}\right)\left(\frac{l_2-1}{\gcd(l_2-1,u_2)}\right)^{-1}$ mod $\frac{u_2}{\gcd(l_2-1,u_2)}$, so for any choice of y' there are $\gcd(l_2-1,u_2)$ choices for x_2 and $\gcd(l_2-1,u_2)$ divides $|C_G(\gamma)|$.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.
- If y' is even, then there are no restrictions on x_2'. There are n ways to choose y' and u_2 ways to choose x_2', so $u_2 n$ divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2, then all choices of x_2' commute with x_2, so u_2 divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If y is odd, then $x_2' \equiv x_2 \left(\frac{l_2^{y'}-1}{\gcd(l_2-1,u_2)}\right)\left(\frac{l_2-1}{\gcd(l_2-1,u_2)}\right)^{-1}$ mod $\frac{u_2}{\gcd(l_2-1,u_2)}$, so for any choice of y' there are $\gcd(l_2-1,u_2)$ choices for x_2 and $\gcd(l_2-1,u_2)$ divides $|C_G(\gamma)|$.
- Because $l_2^2 \equiv 1 \mod u_2$ and u_2 is even, l_2 is odd, so $\gcd(l_2-1,u_2)$ is even and $|C_G(\gamma)|$ is even.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.
- If y' is even, then there are no restrictions on x_2'. There are n ways to choose y' and u_2 ways to choose x_2', so $u_2 n$ divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2, then all choices of x_2' commute with x_2, so u_2 divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If y is odd, then $x_2' \equiv x_2 \left(\frac{l_2^{y'}-1}{\gcd(l_2-1,u_2)}\right)\left(\frac{l_2-1}{\gcd(l_2-1,u_2)}\right)^{-1}$ mod $\frac{u_2}{\gcd(l_2-1,u_2)}$, so for any choice of y' there are $\gcd(l_2-1,u_2)$ choices for x_2 and $\gcd(l_2-1,u_2)$ divides $|C_G(\gamma)|$.
- Because $l_2^2 \equiv 1 \mod u_2$ and u_2 is even, l_2 is odd, so $\gcd(l_2-1,u_2)$ is even and $|C_G(\gamma)|$ is even.
- Therefore, $\Gamma(G)$ is Eulerian.

Semidirect products (continued)

- If y is even, then either y' is even or $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2.
- If y' is even, then there are no restrictions on x_2'. There are n ways to choose y' and u_2 ways to choose x_2', so $u_2 n$ divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If $\frac{u_2}{\gcd(l_2-1,u_2)}$ divides x_2, then all choices of x_2' commute with x_2, so u_2 divides $|C_G(\gamma)|$ and $|C_G(\gamma)|$ is even.
- If y is odd, then $x_2' \equiv x_2 \left(\frac{l_2^{y'}-1}{\gcd(l_2-1,u_2)}\right)\left(\frac{l_2-1}{\gcd(l_2-1,u_2)}\right)^{-1}$ mod $\frac{u_2}{\gcd(l_2-1,u_2)}$, so for any choice of y' there are $\gcd(l_2-1,u_2)$ choices for x_2 and $\gcd(l_2-1,u_2)$ divides $|C_G(\gamma)|$.
- Because $l_2^2 \equiv 1 \mod u_2$ and u_2 is even, l_2 is odd, so $\gcd(l_2-1,u_2)$ is even and $|C_G(\gamma)|$ is even.
- Therefore, $\Gamma(G)$ is Eulerian.
- Therefore, $\Gamma(\mathbb{Z}_{2m} \rtimes \mathbb{Z}_{2n})$ is Eulerian for all positive integers m and odd positive integers n.

Path Eulerian non-commuting graphs

- Out of all groups that have been considered so far, the only one that has had a path Eulerian non-commuting graph is $S_3 \cong D_6$. I conjecture that there is no other group G for which $\Gamma(G)$ is path Eulerian.

Path Eulerian non-commuting graphs

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- Because $[x] = \{x, x^2\} = \{x, x^{-1}\}$, $txt^{-1} = x^{-1}$, so $tx = x^{-1}t$.

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