Characteristics of Non-commuting Graphs of Groups

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Orientable Genus

Definition

Orientable genus of a graph is the minimum number of handles required on a sphere to embed the graph.

- Genus of a graph G is denoted by $\gamma(G)$.

Example

The torus is an orientable genus 1 surface, the two holed torus genus 2 surface, and so on.

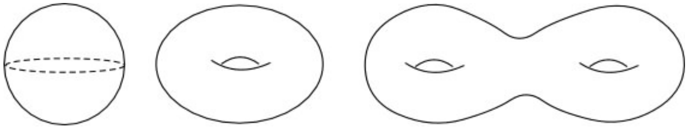

Figure : Genus 0, 1 and 2

Non-Orientable Genus

Definition

Non-orientable genus of a graph represents the minimum number of cross caps required on a sphere to embed the graph.

- The non-orientable genus of a graph G is denoted by $\tilde{\gamma}(G)$.

Example

The projective plane has non-orientable genus 1 and the klein bottle has non-orientable genus 2.

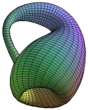

Figure : Klein Bottle

Genus Theorems

Ringel and Youngs

Theorem

For the complete graph K_n, $n \geq 3$

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \quad (1)$$

and if $n \geq 3$ and $n \neq 7$ then

$$\tilde{\gamma}(K_n) = \lceil \frac{(n-3)(n-4)}{6} \rceil \quad (2)$$

.

Genus Theorems

Ringel

Theorem

If $m, n \geq 2$ then for the complete bipartite graph, $K_{m,n}$

$$\gamma(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil \quad (3)$$

and if $m, n \geq 3$ then

$$\tilde{\gamma}(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{2} \rceil \quad (4)$$

Genus Theorems

Kuratowski

Theorem

A graph G is planar or genus 0 if and only if C does not contain a subdivision of K_5 or $K_{3,3}$.

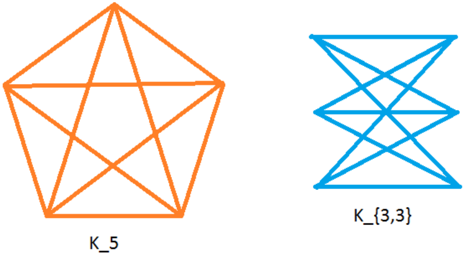

Figure : Graphs of K_5 and $K_{3,3}$

Genus Theorems

Euler

Theorem

For a 2-cell embedding of a graph G on S_γ it holds that if V is the number of vertices of the graph, E the number of edges, and F the number of faces then

$$V - E + F = 2 - 2\gamma \quad (5)$$

$$V - E + F = 2 - \tilde{\gamma} \quad (6)$$

Properties
A. Abdollahi et. al.

- For every connected graph G that is not a tree, $\tilde{\gamma}(G) \leq 2\gamma(G) + 1$.
- If G is a non-abelian group then $\Gamma(G)$ is planar if and only if G is isomorphic to either S_3, D_8, or Q_8.

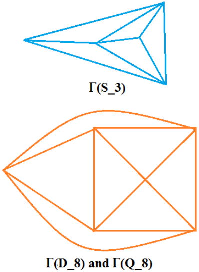

Figure : $\Gamma(S_3) and \Gamma(D_8) \cong \Gamma(Q_8)$

Genus Lower Bound

- We know $2E = 3F_3 + 4F_4 + 5F_5 \dots$ then $2E \geq 3(F_3 + F_4 + \dots)$ therefore $2E \geq 3F$.
- Since $V - E + F = 2 - 2\gamma$ then $\gamma \geq \lceil \frac{E}{6} - \frac{V}{2} + 1 \rceil$
- Similarly for non-orientable genus $\lceil \frac{E}{3} - V + 2 \rceil \leq \tilde{\gamma}$

Genus Upper Bound

- Use genus of graph in which it is embedded
  - Complete graph
  - Complete bipartite graph

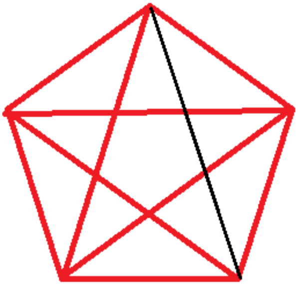

Figure : $\Gamma(D_6)$ shown as a subgraph of K_5

Genus Limiting Search

- We know $d(v) = |G| - |C_G(v)|$.
- Since $C_G(v)$ is a subgroup of G, then $|C_G(v)| \mid |G|$.
  - Therefore $|C_G(v)| = |G|$ or $|C_G(v)| \leq \frac{|G|}{2}$.
  - If $|C_G(v)| = |G|$ then $C_G(v) = G$ so $v \epsilon Z_G$ and v will not belong to $\Gamma(G)$.
  - Hence $|C_G(v)| \leq \frac{|G|}{2}$.
- We see that $1 \leq |C_G(v)| \leq \frac{|G|}{2}$ for all $v \epsilon \Gamma(G)$.

Genus Limiting Search

We are able to set a bound on the lowest vertex degree.

Example

We will show for genus 2 non-commuting graphs of a group, G, has a vertex degree less than or equal to 7 ($d(u) \leq 7$) for some $u\epsilon G/C(G)$.
First, suppose not. Thus every vertes has $d>7$.

- $2E = \sum d(v)>7v$
- $E>\frac{7V}{2}$
- $\gamma>\lceil \frac{7V}{6} - \frac{6V}{12} + 1\rceil = 2$
- So by the previous argument for genus 2 we only need to check groups of $o(G) \leq 14$.

This can be done for higher genus as well.

Finding Genus (Bounds)

Example

Consider $\Gamma(D_{10})$.

- $D_{10} =< a, b | a^5 = b^2 = e, bab^{-1} = a^{-1} >$
- $Z(D_{10}) = \{1\}$
- $E = 30$ and $V = 9$

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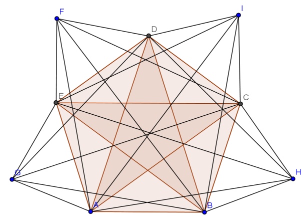

Figure : $\Gamma(D_{10})$

Finding Genus (Bounds)

Example

- Lower Bound: $\gamma(\Gamma(D_{10})) = 2$; Upper Bound: $\gamma(K_9) = 3$

Finding Genus (Bounds)

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- Lower Bound: $\gamma(\Gamma(D_{10})) = 2$; Upper Bound: $\gamma(K_9) = 3$
- Therefore $2 \leq \gamma(\Gamma(D_{10})) \leq 3$

Finding Genus (Bounds)

- We were able to embed the graph on the two holed torus:

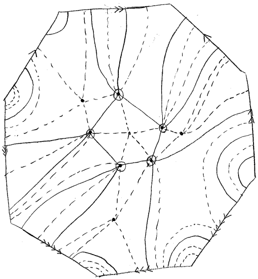

Figure : $\Gamma(D_{10})$ on the 2 holed torus

- Therefore the genus of $\Gamma(D_{10})=2$

Triangle Faces

- Each edge of a graph can only be an edge of two different triangles.
- $2E = 3F_3 + 4F_4 + 5F_5 \ldots$ so consider $2E \geq 3F_3 + 4(F - F_3)$.
- $\gamma \leq \frac{E}{2} - V - \frac{F_3}{4} + 2$
- $F_3 \leq \frac{2E}{3}$ consider $F_3 = \frac{2E}{3}$. This is the smallest the genus can be.
- We use this in cases where if $F_3 = \frac{2E}{3}$ is possible on an orientable surface then the genus of the graph is its lower bound.

Finding Genus (Triangle Representation)

Example

- The Non-orientable genus of $\Gamma(D_{12})$ has boun4 $\leq \tilde{\gamma} \leq 7$

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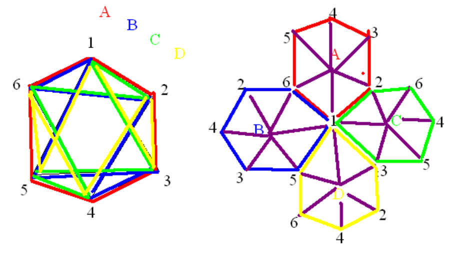

Figure : $\Gamma(D_{12})$ Non-orientable Genus Triangle Representation

Therefore $\tilde{\gamma}(\Gamma(D_{12})) = 4$

Finding Genus (Triangle Representation)

Example

- We have not found a triangle representation that generates a orientable genus representation for non-commuting graphs.
- We know that $\Gamma(D_{10})$ can not be represented using only triangles

Orientable Genus

- For a group G there exist no $\Gamma(G)$ such that the graph is orientable or non-orientable genus 1

Orientable Genus

- For a group G there exist no $\Gamma(G)$ such that the graph is orientable or non-orientable genus 1
- $\gamma(\Gamma(D_{10})) = 2$
- $\gamma(\Gamma(M_{16})) = 3$

Non-Orientable Genus

- There are no non-commuting graphs of groups of non-orientable genus 1 or 2.

Non-Orientable Genus

- There are no non-commuting graphs of groups of non-orientable genus 1 or 2.
- $\tilde{\gamma}(\Gamma(D_{12})) = 4$, $\tilde{\gamma}(\Gamma(D_{16})) = 12$, and $\tilde{\gamma}(\Gamma(D_{20})) = 24$.

Genus

Continued Research

- Find all non-commuting graphs of groups with non-orientable genus 3.

Genus

Continued Research

- Find all non-commuting graphs of groups with non-orientable genus 3.
- Conjecture: $\tilde{\gamma}(\Gamma(D_{2n})) = \lceil \frac{n^2}{2} - 3n + 4 \rceil$ for n even.
- Conjecture: $\gamma(\Gamma(D_2 n)) \neq \lceil \frac{n^2}{4} - \frac{5n}{4} + \frac{3}{2} \rceil$ for n odd.

Genus

Continued Research

- Find all non-commuting graphs of groups with non-orientable genus 3.
- Conjecture: $\tilde{\gamma}(\Gamma(D_{2n})) = \lceil \frac{n^2}{2} - 3n + 4 \rceil$ for n even.
- Conjecture: $\gamma(\Gamma(D_2 n)) \neq \lceil \frac{n^2}{4} - \frac{5n}{4} + \frac{3}{2} \rceil$ for n odd.
- Determine which graphs can be represented using only triangles.
- Work on equations to find genus of non-commuting graphs of families of groups.

Chromatic Number

Definition

The chromatic number of a graph G is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices are colored the same.

- Chromatic Number of a graph G is denoted $\chi(G)$.

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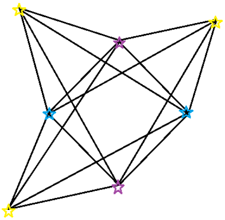

Figure : Chromatic Coloring of $\Gamma(D_8)$

Chromatic Polynomial

Definition

The chromatic polynomial of a graph G is a function $P(G,t)$ that counts the number of proper vertex colorings using t colors on graph G.

- The chromatic number is the minimum value of t for which $P(G,t)$ is non zero.
- Two graphs, G and H, can share a common chromatic polynomial without $G \cong H$.

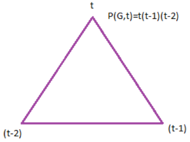

Figure : Chromatic coloring of K_3 with t colors.

Chromatic Number

- $\lambda(\Gamma(D_{2n})) = n + 1$ for n odd.
- $\lambda(\Gamma(D_{2n})) = \frac{n}{2} + 1$ for n even.
- $\lambda(\Gamma(G_{p,q,k})) = p + 1$ for p,q prime and $k^q \equiv 1 mod p$.

Chromatic Polynomial

- $P(\Gamma(D_{2n})) = t(t-1)(t-2)...(t-(n-1))(t-n)^{n-1}$ for n odd.
- $P(\Gamma(D_{2n})) = t(t-1)\ldots(t-(n-1))\binom{n}{0}(t-n)^{n-2} + t(t-1)\ldots(t-n)\binom{n}{1}(t-(n+1))^{n-2} + \cdots + t(t-1)\ldots(t-(2n-1))\binom{n}{n}(t-2n)^{n-2}$

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- General: $P(\Gamma(K_{2,2,\dots,2,m})) = t(t-1)\dots(t-(n-1))\binom{n}{0}(t-n)^{m} + t(t-1)\dots(t-n)\binom{n}{1}(t-(n+1))^{m} + \dots + t(t-1)\dots(t-(2n-1))\binom{n}{n}(t-2n)^{m}$

Chromatic Colorings

Continued Research

- Look into the chromatic number and chromatic polynomial for more families of graphs.
    - Specifically k-partite graphs
- Analyze what the chromatic number of non-commuting graphs tells us about the groups.

Cop Number

Definition

The cop number on a graph G (denoted $c(G)$)is the least number of cops required to have a winning strategy in cops and robbers on the graph.

- Cops and robbers is played by first placing n number of cops on vertices and 1 robber at a vertex. Then the cops attempt to move to the vertex that the robber occupies by alternating turns where the cops move (or chose not to move) and then the robber moves (or choses not to move). Movement can only take place along an edge of the graph.

Cop Number

Consider $\Gamma(D_6)$

Example

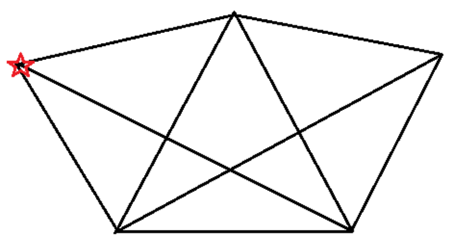

Figure : Move 1

Cop Number

Example

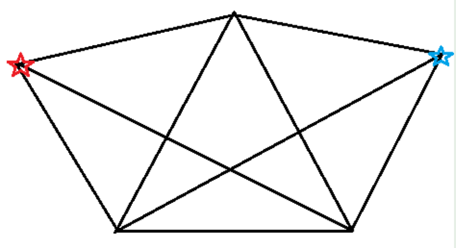

Figure : Move 2

Cop Number

Example

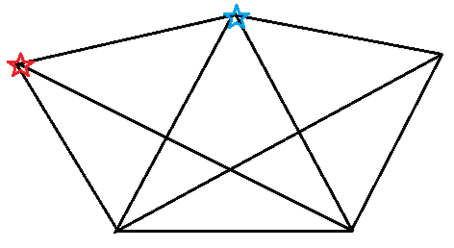

Figure : Move 3

Cop Number

Example

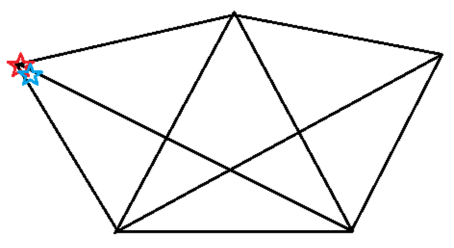

Figure : Move 4

Now the robber is caught.

Cop Number

Example

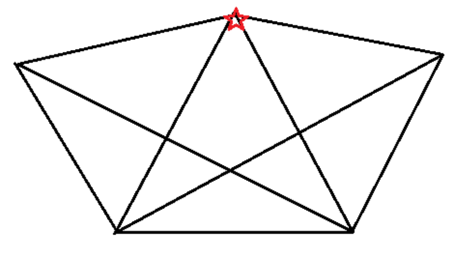

Figure : Move 1

Cop Number

Example

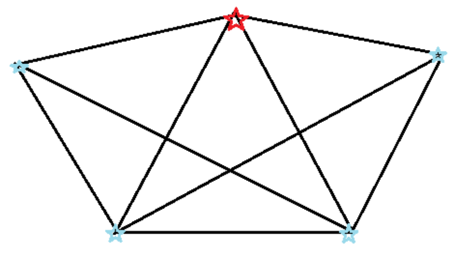

Figure : Move 2

Now the robber is caught. So $c(\Gamma(D_6)) = 1$

Cop Number

Example

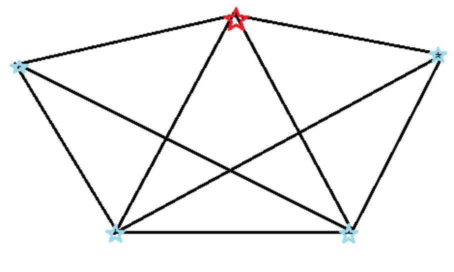

Figure : Move 2

Now the robber is caught. So $c(\Gamma(D_6)) = 1$

Cop number

- $c(\Gamma(D_{2n})) = 1$ for n odd.
- $c(\Gamma(D_{2n})) = 2$ for n even.
- $c(K_{\alpha_1,\alpha_2,\ldots}) \leq 2$.

Cop Number

Continued Research

- Look at characteristics of the cop number across different families of groups for non-commuting graph.
- Determine similarities among groups with non-commuting graphs of the same cop number.

Graph Isomorphism

Definition

Two graphs Γ_1 and Γ_2 are said to be isomorphic if there exists a bijection f between their vertex sets and x and y are connected by an edge if and only if $f(x)$ and $f(y)$ are connected by an edge.

- Knowing which groups have isomorphic non-commuting graphs helps us narrow our computation for graph properties
- If two groups are isomorphic we know they share a common chromatic number, cop number, characteristic polynomial, genus, etc.

Graph Isomorphism

Abdollahi et. al. Conjecture

- In 2006, Abdollahi et. al. made the conjecture "Let G and H be two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$."

Graph Isomorphism

Abdollahi et. al. Conjecture

- In 2006, Abdollahi et. al. made the conjecture "Let G and H be two non-abelian finite groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$."
- This has been proven false using groups of order $|G| = 2^{10}5^3 \neq 2^3 5^6 = |H|$.
- So far it is known the conjecture is true if G is a dihedral group, alternative group, or non-solvable AC-group.

Graph Isomorphism

- The following holds:
  - $\Gamma(M_{p^n}) \cong K_{(p-1)p^{n-2},(p-1)p^{n-2},\ldots,(p-1)p^{n-2}<p+1 times>}$ for p prime.

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  - $\Gamma(D_{2^n}) \cong \Gamma(Q_{2^n})$.

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  - $\Gamma(D_{2^n}) \cong \Gamma(Q_{2^n})$.
  - $\Gamma(M_{16}) \cong \Gamma(Z_4 \rtimes Z_4) \cong \Gamma(SU(2)) \cong \Gamma(Q_8 x Z_2) \cong \Gamma((Z_2 x Z_2) \rtimes Z_4) \cong \Gamma(D_8 x Z_2)$.

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  - $\Gamma(D_{2n}) \cong K_{(n-1),1,1,\ldots(n1's}$ for n odd.
  - $\Gamma(D_{2n}) \cong K_{(n-2),2,2,\ldots(\frac{n}{2}2's)}$ for n even.

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  - $\Gamma(D_{2n}) \cong K_{(n-1),1,1,\ldots(n1's}$ for n odd.
  - $\Gamma(D_{2n}) \cong K_{(n-2),2,2,\ldots(\frac{n}{2}2's)}$ for n even.
  - $\Gamma(G) \cong K_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ for G an AC group and $1 \leq \alpha_i$.

Graph Isomorphism

Continued Research

- Verify if G and H are non-abelian groups where $|G| = |H|$ and $\Gamma(G)$ and $\Gamma(H)$ have the same number of edges and vertices then $\Gamma(G) \cong \Gamma(H)$.
  - May or may not hold for larger groups

Graph Isomorphism

Continued Research

- Verify if G and H are non-abelian groups where $|G| = |H|$ and $\Gamma(G)$ and $\Gamma(H)$ have the same number of edges and vertices then $\Gamma(G) \cong \Gamma(H)$.
  - May or may not hold for larger groups
- Look for similarities among groups with isomorphic non-commuting graphs.

Graph Isomorphism

Continued Research

- Verify if G and H are non-abelian groups where $|G| = |H|$ and $\Gamma(G)$ and $\Gamma(H)$ have the same number of edges and vertices then $\Gamma(G) \cong \Gamma(H)$.
  - May or may not hold for larger groups
- Look for similarities among groups with isomorphic non-commuting graphs.
- Characterize groups in which if G and H are non-abelian groups such that $\Gamma_G \cong \Gamma_H$ then $|G| \neq |H|$.

Summary

- There exist no genus 1 non-commuting graphs and no non-orientated genus 1 or 2 non-commuting graphs.
- The non-commuting graphs of D_{2n} are determined completely for cop number, graph isomorphism, chromatic number, characteristic polynomial, and chromatic polynomial.
- We know the isomorphisms between graphs for D_{2n}, M_{p^n}, and $G_{p,q,k}$.

- Outlook
  - Define D_{2n} orientable and non-orientable genus for n odd and n even.
  - Finding further strategies for computing genus and determine isomorphisms between families of groups non-commuting graphs.
  - Identify chromatic polynomials for more families of graphs.

Table of Results for degree ≤ 22

Graph ID	γ	$\tilde{\gamma}$	λ	c	$\cong$
S_3	0	0	4	1	$K_{2,1,1,1}$
D_8	0	0	3	2	$K_{2,2,2}$
D_{10}	2	$4 \leq \gamma \leq 5$	6	1	$K_{4,1,1,1,1,1}$
D_{12}	$2 \leq \gamma \leq 4$	4	4	2	$K_{4,2,2,2}$
A_4	$4 \leq \gamma \leq 5$	$3 \leq \tilde{\gamma} \leq 10$	5	2	$K_{3,2,2,2,2}$
D_{14}	$5 \leq \gamma \leq 8$	$10 \leq \tilde{\gamma} \leq 12$	8	1	$K_{6,1,1,1,1,1,1,1}$
D_{16}	$6 \leq \gamma \leq 10$	12	5	2	$K_{2,2,2,2,6}$
M_{16}	3	$6 \leq \tilde{\gamma} \leq 7$	3	2	$K_{4,4,4}$
D_{18}	$11 \leq \gamma \leq 16$	$21 \leq \tilde{\gamma} \leq 31$	10	1	$K_{8,1,1 \ldots (9\ 1's)}$
$D_6 x \mathbb{Z}_3$	$7 \leq \gamma \leq 11$	$14 \leq \tilde{\gamma} \leq 22$	4	2	$K_{6,3,3,3}$
D_{20}	$12 \leq \gamma \leq 18$	24	6	2	$K_{2,2,2,2,2,8}$
$Z_5 \rtimes Z_4$	$17 \leq \gamma \leq 20$	$33 \leq \tilde{\gamma} \leq 40$	6	2	$K_{4,3,3,3,3,3}$
$Z_7 \rtimes Z_3$	$19 \leq \gamma \leq 23$	$38 \leq \tilde{\gamma} \leq 46$	8	2	$K_{6,2,2,2,2,2,2,2}$
D_{22}	$18 \leq \gamma \leq 26$	$36 \leq \tilde{\gamma} \leq 51$	12	1	$K_{10,1,1,\ldots (11\ 1's)}$

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Bibliography I

B. Mohar, C. Thomassen.
Graphs on Surfaces.
The John Hopkins University Press, 2001.

A. Abdollahi, S. Akbari, H.R. Maimani.
Non-commuting graph of a group.
Journal of Algebra, 02(15):468–492, 2006.

A. Talebi.
Non-commuting graph of a group D_{2n}.
International Journal of Algebra, 02(20):957–961, 2008.