On Fractals and Dimensionality

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What the fractals are these?

Benoit Mandelbrot coined the term ‘Fractal’ from the Latin *fractus*, meaning broken, in order to describe (mathematical) objects that were too irregular to fit into a traditional geometric setting. There is a notion of self-similarity associated with fractals, i.e. fractals tend to appear more or less the same regardless of the magnification imposed on them.

Mathematical definition

A fractal is a mathematical object whose Hausdorff dimension is strictly greater than its topological dimension.

...What the fractal?

Topological dimension (reference (page 9) Topological dimensions, Hausdorff dimensions by Kohavi and Davdovich): The topological dimension of a topological space X is defined to be the minimum value n, such that any open cover has a refinement in which no point is included in more than $n + 1$ elements. If such an n doesn't exist, then X is of infinite topological dimension.

Intuitively speaking, this is the dimension of the boundary of X plus 1. For example, the intuitive geometric definition of a $k - cube$ in $\mathbb{R}^k$ is k, and the intuitive geometric definition of its boundary is $k - 1$. Thus, its topological definition will be the dimension of its boundary plus 1, i.e. $(k - 1) + 1 = k$.

It is clear that the notion of the topological dimension is an intuitive one. Some basic properties include:

- The dimension of a space X is -1 iff $X = \emptyset$.
- The dimension of X is $\leq n$ if for every point $x \in X$ and for every open set P, there exists an open Q such that $\overline{P} \subset Q$ and the dimension of ∂P is $\leq n - 1$, where ∂P denotes the boundary of P.
- The dimension of X is $= n$ if the above holds true for n, but not for $n - 1$.
- The dimension of X is $= \infty$ if, for every $n \in \mathbb{N}$, the dimension of X $\leq n$ doesn't hold.

Hausdorff dimension (page 12-13): The Hausdorff definition is a slightly more complicated matter than the topological definition, so we'll build up to it's definition with some useful concepts. However, it is important to note that the Hausdorff dimension is only defined for metric spaces.

Drumroll

Let X be a metric space and C a collection of subsets. The **mesh** of C is $mesh(C) = \sup\{diam(B) : B \in C\}$.

Drumroll

Let $A \subset X$. We denote
$(H_d)(A, \epsilon) = \inf\{\sum(diam((B_i)^d)) : C = \{B_i\}, mesh(C) < \epsilon\}$, where C is a countable cover of A.

Drumroll

We define the Hausdorff p-measure M_d to be $M_d(A) = \sup\{H_d(A, \epsilon)\}$.

Drumroll

This definition is equivalent to $M_d(A) = \lim_{\epsilon \to 0} H_d(A, \epsilon)$.
This measure "tells" the "volume" of A as if A were in a d-dimensional space.

Isn't this a rather long drumroll

Yes, yes it is.

Ta-da!

We define the Hausdorff dimension of A by
$dim_H(A) = \sup\{d : 0 < M_d(A) < \infty\}$.
This is a very natural definition, as it gives the highest dimension in which A has a finite, nonzero (if possible) "volume". For example, the "volume" of a square in 1 dimension would be ∞ and in 3 dimensions would be 0. However, in two dimensions, it would be given by the square of its sides.

But that's obvious!

Of course, the topological dimension of the square matches its Hausdorff dimension, which is pretty boring. So we introduce the hipsters of geometry, i.e. **fractals**, as objects whose Hausdorff dimension is strictly different than their topological dimension.

Hipsters of Geometry

Fractals: Actually, we prefer being called fractals. “Hipster” is too mainstream.

Types of Fractals

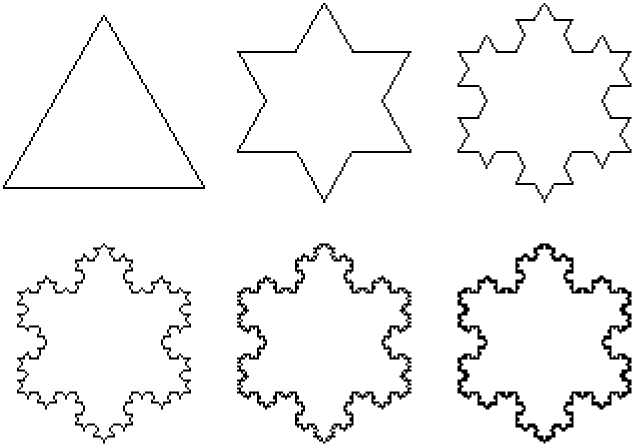
There are many classes of Fractals, primarily based on how they are generated. Many of them fall into the category of "Iterated Function Systems" or I.F.S. As the name suggests, this technique involves repeated application of the same function(s) to a given set, usually in the form of a contraction mapping. Some notable types include:

Base-Motif Fractals:
This method of generating fractals involves starting with a base, which could be any composition of line segments, and then replacing the base with a "motif", i.e. a transformation applied to the base. For example: The Koch snowflake has a triangular base, and, to each line segment, the motif is to add a triangle in the middle.

Base

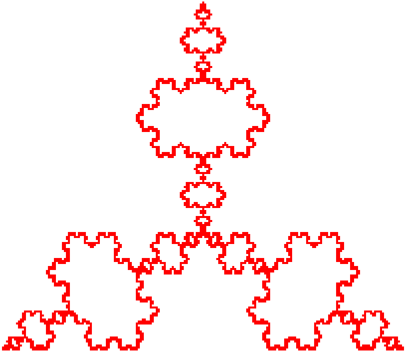
Motif

The Koch snowflake

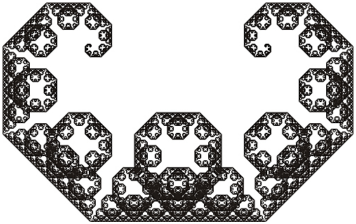


Similarly, note that the orientation matters, as orienting the motif towards the inside gives the Koch Anti-snowflake:

The Koch Anti-snowflake

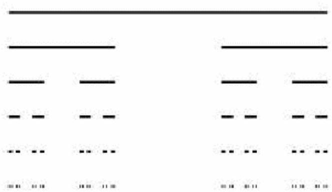


Other Base-Motifs



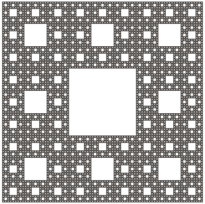
Dusts and Clusters: This is a specific type of base-motif which involves removal. For one dimensional objects, repeatedly removing certain parts of the objects will result in a "dust" fractal. A very famous example is, of course, the Cantor set:

Cantor Set



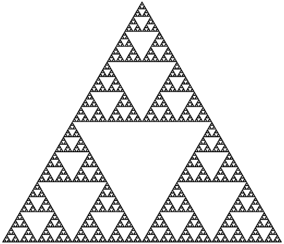
Similarly, when the same is done for two or more dimensional objects, the resulting fractal is called a “cluster” fractal. For example, the Sierpinski Carpet:

Carpet



And the Sierpinski Pyramid:

Pyramid



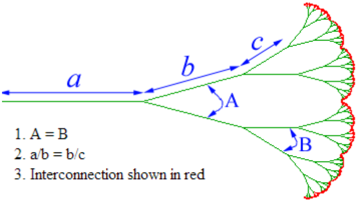
Types of Fractals

Fractal Canopies: Fractal canopies are fractals formed by branching so that certain criteria are fulfilled. For example, if the starting object is a line segment, then the resulting fractal canopy must satisfy the following properties:

Properties:

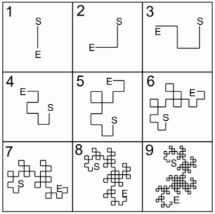
- The angle between any two neighbouring line segments is the same throughout the fractal.
- The ratio of lengths of any two consecutive line segments is constant
- The points towards the end (i.e. the direction in which the canopy grows) of the smaller line segments should eventually be interconnected.

Canopy

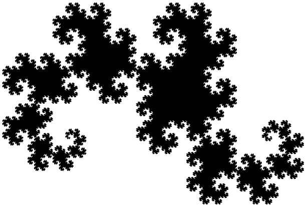


Paper-Folding Fractals: These are formed from “folding” a long strip of paper, but are essentially just another form of base-motif fractals. Consider a line representing the long strip of paper, and suppose the strip is folded and unfolded so that the angle between the two halves is 90 degrees (The angle could really be anything). To each half, do the same. The fractal thus formed (Specifically with 90 degrees) is called the dragon fractal:

Dragon

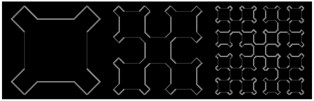


Dragon

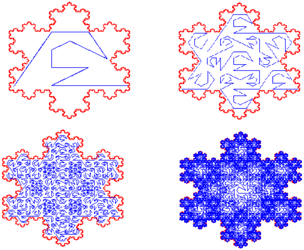


Space-filling Curves: Space filling curves are so named because their Hausdorff dimension exceeds their topological dimension by an integer, and hence seem to "fill (bigger) space" than one may think possible. For example, the Sierpinski curve can completely "fill" a square (topological dimension 2), even though its topological dimension is 1. Space filling curves can generally fill any shapes, including other fractals! For example, the Snowflake sweep can fill the Koch snowflake.

Curve



Snowflake Sweep



Mandelbrot

Types of Fractals

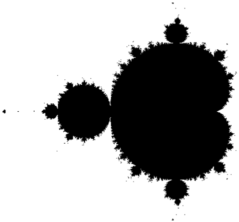
Mandelbrot Sets: French mathematician Benoit Mandelbrot (known in pop culture as Yoda) was the first to thoroughly study and popularize fractals. In his honour, a class of fractals was named "Mandelbrot sets". In general, Mandelbrot sets are a specific type of Julia sets, which are another class of more complicated fractals, which are (topologically) connected.

Mandelbrot

True, it is.

"The" Mandelbrot set is defined by the quadratic recurrence equation: $z_{n+1} = {z_n}^2 + C$, where $z_0 = C$, where the points $C \in \mathbb{C}$ such that the orbit of z_n does not tend to infinity. That is, the admissible C-values are exactly those which give a finite value for **every** iteration in the recurrence equation given above. Eg. If $C = 1$, then $z_1 = 2$, $z_3 = 5$, $z_4 = 26$, and so on, which tends to infinity. So 1 is not an element of the Mandelbrot set. However, $C = -1$ is bounded between 0 and -1, so it would be an element of the Mandelbrot set.

Mandelbrot



Similitudes and Calculating Dimensions

Similitudes: These are a type of affine transformations used to generate fractals with (generally) a scaling factor, a rotation matrix, and a constant term. In order to create fractals using similitudes, we begin with a self similar set, and apply the similitude to the set, which transforms each point in the set according to the equation at each stage.

Similitudes

General form of a similitude:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix},$$

where a, b, c, θ are scalars.

Similitudes

Here's how to use similitudes to generate the Sierpinski carpet, beginning in a square with opposite corners $(0,0)$ and $(1,1)$: The 8 similitude equations are:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$$

Similitudes

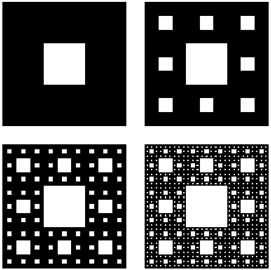
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{2}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

S-Carpet



Calculating dimensions

As we saw before, the technical definition of Hausdorff dimension is rather messy, and hence not the most efficient in determining dimensions of certain "nice" fractals, i.e. fractals with a high degree of self similarity. For many "not nice" fractals, a precise way of actually calculating their dimensions is not known, even though we have the formal definition of Hausdroff dimension. For such cases, the best we can do is to get good estimates for bounds. Fortunately, there is a basic model for finding (precisely) the dimension of nice fractals, which is far less complicated.

Calculating dimensions

Basic model: Recall that $n = \frac{1}{s^D}$, where D is the dimension, s is the scaling factor, and n is the number of copies of smaller fragments of the fractal. This equation comes from the observation that $n * s^D = 1$, i.e. the number of copies of smaller fragments of a set times the scaling factor gives back the set itself.

Calculating dimensions

Since $n = \frac{1}{s^D}$, we have:

$$\begin{aligned}
n &= \frac{1}{s^D}, \\
log(n) &= log(\frac{1}{s^D}), \\
log(n) &= D * log(\frac{1}{s}), \\
D &= \frac{log(n)}{log(1/s)}
\end{aligned}$$

Calculating dimensions

Example: The Koch curve: The number of line segments in the Koch curve is 4, and each line segment is replaced by a copy of the original reduced by a scale of $1/3$. Thus, the dimension of the Koch curve is $\frac{log(4)}{log(3)} \approx 1.26$.

Koch Curve

Until the 19th century, it was thought that any function with an analytic formula, i.e. sum of a convergent power series would produce a differentiable curve. However, Karl Weierstrass begged to differ when he presented a paper to the Royal Prussian Academy of Sciences showing that for a positive integer a and $0 < b < 1$, the function

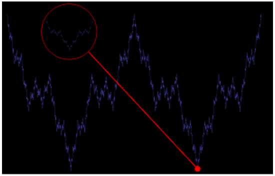
$$\sum_{n=1}^{\infty} b^n cos(a^n x\pi)$$

is nowhere differentiable.

The Weierstrass function

Weierstrass proved that the previous function was nowhere differentiable while being continuous. Now, the Weierstrass function is itself a fractal, and the idea behind the proof of it being nowhere differentiable relies on properties attributed to it being a fractal.

The Weierstrass function



The Weierstrass function

The basic underlying geometric idea is that for a differentiable function, one can “zoom in” on particular parts of the curve so that they would resemble a straight line, i.e. it is possible to make tangent lines (or tangent planes, for higher dimensional curves) on it. However, zooming in has no such effect on a fractal, as every part of the fractal will remain as jagged as it was before. From this idea, we can deduce that fractals are nondifferentiable.

References

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- Klang, Jesse. "An Introduction to Fractals." An Introduction to Fractals. N.p., n.d. Web.

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