# Convergence of a Generalized Midpoint Iteration

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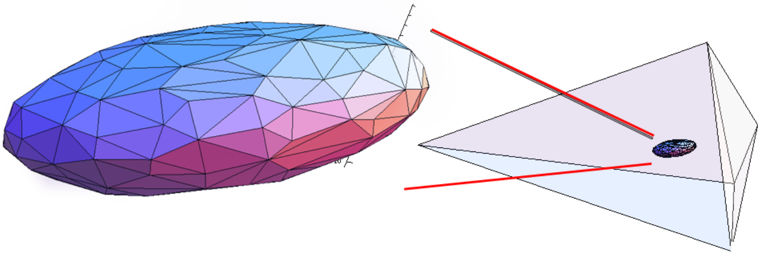
# O'Rourke's conjecture

We begin with a motivating question concerning the convergence of sequences of centroid polyhedra.

## Conjecture

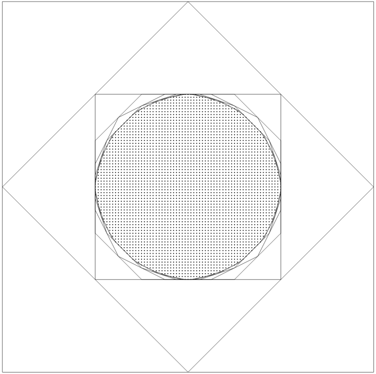
*Given any convex polyhedron $P^{(0)}$ in $\mathbb{R}^3$, the iteration process which takes the convex hull $c(P^{(n)})$ of the centroids of faces of $P^{(n)}$ converges to an ellipsoid as $n \to \infty$.*

## Preliminary Words



# O'Rourke's conjecture

To work towards a solution of O'Rourke's conjecture, we examine a simpler iteration of midpoints in $\mathbb{R}^2$ similar to the midpoint iteration of convex polygons.



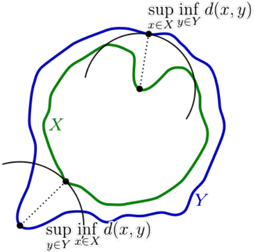
# Notation

- Let $d(\cdot,\cdot)$ denote the Euclidean distance between two points.
- Let $|\cdot|$ denote the Euclidean norm.
- Let $c(\cdot)$ denote the convex hull of a set.
- Let $cl(\cdot)$ denote set closure (closure includes limit points).
- Let $B(x,\varepsilon)$ denote the open ball of radius $\varepsilon$ about $x$.
- Let $diam(\cdot)$ denote the diameter of a set.

# Hausdorff distance

For two nonempty compact subsets $X, Y$ of $(\mathbb{R}^2, d)$, we denote the Hausdorff distance by

$$d_H(X, Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(y,x)\right\}.$$



# Propositions about compact sets

We identify a polygon $P$ with the convex hull of a finite set of vertices in $\mathbb{R}^2$. Recall the Heine-Borel characterization of compact subsets of $\mathbb{R}^2$ (compact iff bounded and closed). Then $P$ is a bounded and closed set of the plane.

There are many useful results about nested sequences of compact subsets which can be applied to polygons on the plane.

# Propositions about compact sets

## Proposition

*Suppose $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$, where all $K_n$ are compact, nonempty sets. Then the intersection $I = \bigcap_{n\in\mathbb{N}} K_n$. is nonempty and compact.*

*Proof.* $I$ is closed and bounded, since arbitrary intersections of closed sets are closed and $I \subset K_1$.

Define a sequence $\{a_p\}$ such that $a_p \in K_p \setminus K_{p-1}$. Then there exists by compactness a subsequence $\{a_{n_k}\}$ with limit in $I$. ∎

# Propositions about compact sets

## Proposition

*Let* $I = \bigcap_{n \in \mathbb{N}} K_n$ *be the intersection of* $K_n$. *If* $\lim_n diam(K_n) = 0$, *then* $I = \{x_0\}$ *for some* $x_0 \in K_1$.

*Proof.* If $I$ is not a singleton, then $\operatorname{diam}(I) > 0$. But $I \subset K_n$ for all $n$, implying $\lim \operatorname{diam}(K_n) > 0$. ∎

# Propositions about compact sets

## Proposition

*Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$, and $I$ their infinite intersection $\bigcap_{n\in\mathbb{N}} K_n$. Then*

$$\lim d_H(K_n, I) = 0.$$

# Proof.

By compactness, choose sequences $\{x_n\}, \{y_n\}$ s.t. $x_n \in K_n \setminus I$ and $y_n \in I$ for all $n \in \mathbb{N}$, such that $\{x_n\} \to \alpha, \{y_n\} \to \beta$ and

$$d(x_n, y_n) = \sup_{x \in K_n} \inf_{y \in I} d(x, y).$$

By a contradiction, $\alpha \in I$, so that

$$d(x_n, y_n) = \inf_{y \in I} d(x_n, y) \leq d(x_n, \alpha)$$

And we know:

$$d(x_n, \alpha) \to 0 \text{ as } n \to \infty$$

■

# Propositions about compact sets

## Proposition

*Suppose* $\{A_n\}$ *is a sequence of compact sets such that* $A_n \subset A_{n+1}$ *and* $A_{n\in\mathbb{N}} \subset K$ *for all* $n \in \mathbb{N}$ *and some compact* $K$*. Let* $U = \bigcup_n A_n$*. Then* $\lim A_n = cl(U)$*.*

*Proof.* Similar to previous proof concerning intersections. Identify convergent subsequences of relevant sequences and bound the Hausdorff distance between the sequences so that this distance approaches 0. ∎

# Background

First solved by the French mathematician J.G. Darboux, the midpoint polygon problem has been examined using diverse techniques, including finite Fourier analysis and matrix products [1] [2] [3] [4].

We present an elementary analysis proof of the midpoint polygon problem.

# Proposition

The original problem statement from the American Mathematical Monthly reads

## Proposition

*Let $\Pi = P_0 P_1 \cdots P_{k-1}$ be a closed polygon in the plane $(k \geq 2)$. Denote by $P'_0, P'_1, \ldots, P'_{k-1}$ the midpoints of the sides $P_0P_1, P_1P_2, \ldots, P_{k-2}P_{k-1}$, respectively, obtaining the fist derived polygon $\Pi' = P'_0 P'_1 \cdots P'_{k-1}$. Repeat the same construction on $\Pi'$ obtaining the second derived polygon $\Pi''$, and finally, after n constructions, obtain the nth derived polygon $\Pi^{(n)} = P_0^{(n)} P_1^{(n)} \cdots P_{k-1}^{(n)}$. Show that the vertices of $\Pi^{(n)}$ converge, as $n \to \infty$, to the centroid of the original points $P_0, P_1, \ldots, P_{k-1}$.*

# Midpoint problem

In the language of elementary analysis and Hausdorff convergence,

## Proposition

*Given an initial set of vertices*

$$P^{(0)} = \left\{ v_1^{(0)}, \ldots, v_Q^{(0)} \right\}$$

*in general linear position, the sequence of convex hulls* $c(P^{(n)})$ *produced by midpoint iteration converge in the Hausdorff metric to the centroid of* $c(P^{(0)})$*.*

*Proof.* Without loss of generality, suppose the centroid of $P^{(0)}$ is the origin. The Hausdorff limit $\lim P^{(n)}$ is equal to the intersection $\bigcap_{n \in \mathbb{N}} c(P^{(n)})$, which we know is nonempty and compact.

# Midpoint problem

Generally, the edge lengths are bounded:

$$\sup_i |v_i^{(n)} - v_{i+1}^{(n)}| \leq (\frac{1}{2})^{n-1} \mathrm{diam}(c(P^{(0)}))$$

And the perimeter bounds the diameter of a closed convex polygon, so

$$\mathrm{diam}(c(P^{(k)})) \leq \sum_{i=1}^{Q} |v_i^{(n)} - v_{i+1}^{(n)}| \leq Q(\frac{1}{2})^{n-1} \mathrm{diam}(c(P^{(0)})).$$

Conclude $\lim \mathrm{diam}(c(P^{(n)})) = 0$. Then $\lim c(P^{(n)}) = \{x_0\}$ for some $x_0$. But by calculation, we see that the centroid of $P^{(n+1)}$ is identically the centroid of $P^{(n)}$, the origin. So $0 \in \bigcap c(P^{(n)})$ is the limit of $\{c(P^{(n)})\}$ ■

# Regularity in the limit

## Proposition

*Let $e_j^{(n)}$ denote the jth edge length $|v_j^{(n)} - v_{j+1}^{(n)}|$ of the nth iteration of the initial polygon of Q vertices. Then*

$$\lim_n |e_i^{(n)} - e_j^{(n)}| = 0 \text{ for any } i, j \text{ indices modulo } Q.$$

*Proof.*

$$|e_i^{(n)} - e_j^{(n)}| \leq |e_i^{(n)}| + |e_j^{(n)}| \leq 2 \operatorname{diam}(P^{(n)})$$

since every edge is bounded by the diameter. Now $\lim \operatorname{diam}(c(P^{(n)})) = 0$, so $\lim_n |e_i^{(n)} - e_j^{(n)}| = 0$. ■

# Regularity in the limit

By the previous proposition, given $\varepsilon > 0$, there is a large enough $N$ such that $n > N$ implies

$$|e_i^{(n)} - e_j^{(n)}| < \varepsilon$$

for all $i, j$ indices taken modulo $Q$.

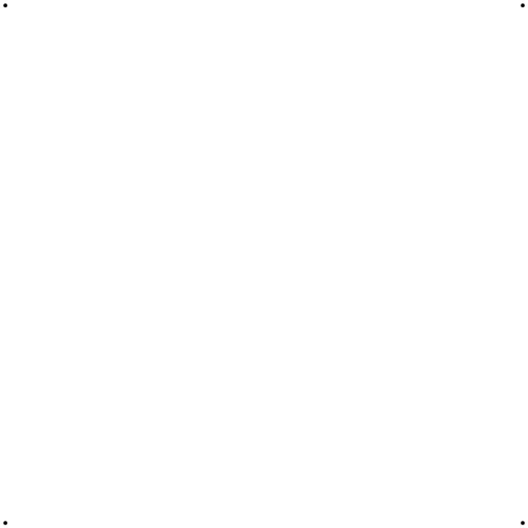
# Generalized Midpoint Iteration

Instead of restricting to midpoints between adjacent vertices, we now consider midpoints of every pair of vertices. This resembles the O'Rourke iteration of centroids of faces of polyhedra because each iteration contains more points than the previous. We will find that this process can be characterized by "fixed" points.

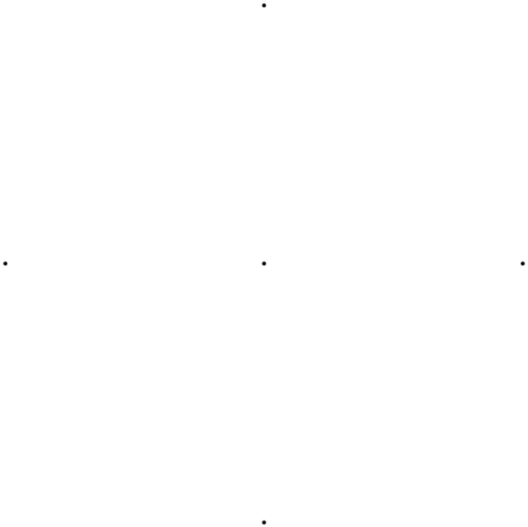
# Generalized Midpoint Iteration

Informally, the *Generalized Midpoint Iteration* (GMI) on $P^{(0)}$ produces vertex sets $P^{(k)}$ containing every possible midpoint combination of the previous vertex set $P^{(k-1)}$, and records how many times $p \in P^{(k)}$ occurs as a midpoint by the multiplicity function $\mu_k : P^{(k)} \to \mathbb{N}$.

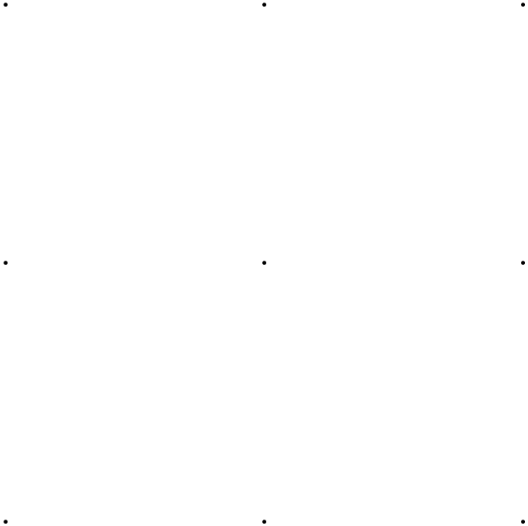
## Generalized Midpoint Iteration



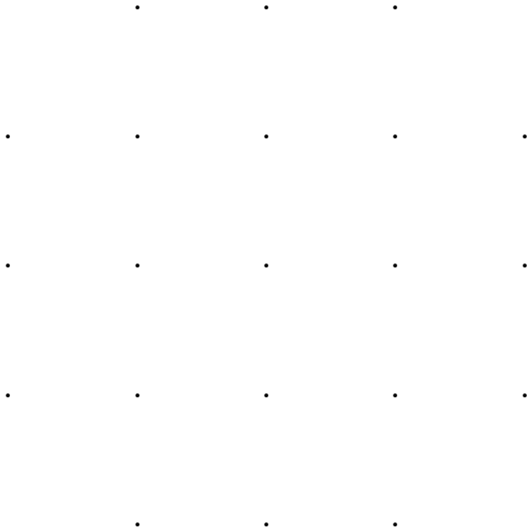
## Generalized Midpoint Iteration

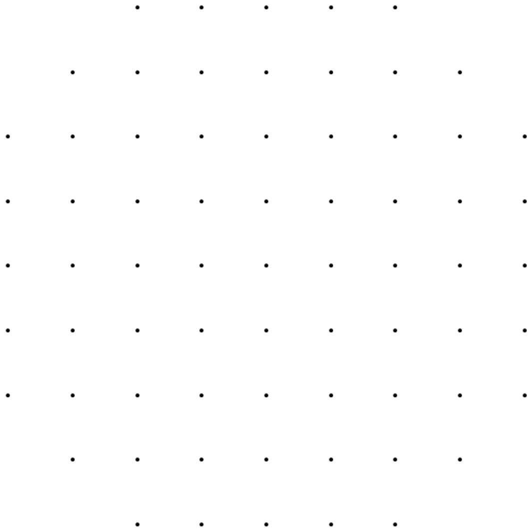


## Generalized Midpoint Iteration



## Generalized Midpoint Iteration





# Subsystems

Within the GMI, we can consider individual sets of three points and their direct descendants for all iterations. The centroid of these subsystems will be in the limit by the classical midpoint iteration on a polygon. This tells us that that GMI does not converge to a single point like the classical midpoint iteration.

# Fixed points

A point is *fixed* if it remains in all succeeding iterations. Under the GMI, a point becomes fixed if it has multiplicity greater than 2 (since there are three ways of pairing a multiplicity 3 point with itself).

Interestingly, we can guarantee, under the GMI on polygons with more than 3 vertices, the existence of explicit fixed points past the second iteration.

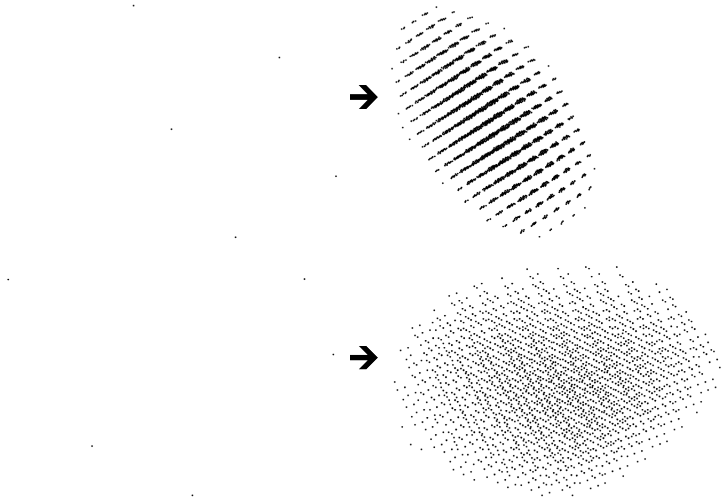
Also, since the midpoint of a fixed point with any other point is fixed (has multiplicity greater than 2), we can eventually "fill up" the object of convergence through one initial fixed point.

# Fixed points

Given any four points, we can find a fixed point of multiplicity 3 after just two iterations using these combinations:

$$\frac{1}{4}\left(3p_1^{(0)}+2p_2^{(0)}+2p_3^{(0)}+p_4^{(0)}\right)$$
$$=\frac{1}{2}\left(\frac{1}{2}(p_1^{(0)}+p_2^{(0)})+\frac{1}{2}(p_1^{(0)}+p_3^{(0)}))+\frac{1}{2}(\frac{1}{2}(p_1^{(0)}+p_2^{(0)})+\frac{1}{2}(p_3^{(0)}+p_4^{(0)})\right)$$
$$=\frac{1}{2}\left(\frac{1}{2}(p_1^{(0)}+p_2^{(0)})+\frac{1}{2}(p_1^{(0)}+p_4^{(0)}))+\frac{1}{2}(\frac{1}{2}(p_1^{(0)}+p_3^{(0)})+\frac{1}{2}(p_2^{(0)}+p_3^{(0)})\right)$$
$$=\frac{1}{2}\left(\frac{1}{2}(p_1^{(0)}+p_2^{(0)})+\frac{1}{2}(p_2^{(0)}+p_3^{(0)}))+\frac{1}{2}(\frac{1}{2}(p_1^{(0)}+p_3^{(0)})+\frac{1}{2}(p_1^{(0)}+p_4^{(0)})\right)$$

# GMI Examples

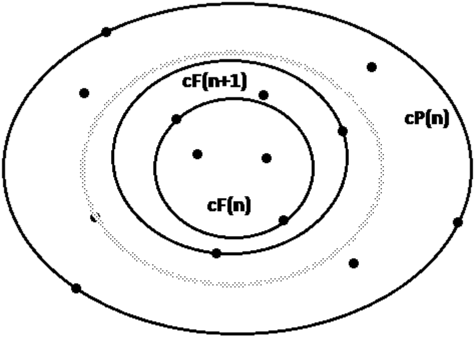


# Goal

We now prove several convergence results involving the sets of fixed points $F^{(n)}$, iteration midpoints $P^{(n)}$, and their convex hulls $c(F^{(n)})$ and $c(P^{(n)})$ respectively.

$$\lim d_H(F^{(n)}, c(F^{(n)})) = 0$$
$$\lim d_H(P^{(n)}, F^{(n)}) = 0$$
$$\lim d_H(P^{(n)}, c(F^{(n)})) = 0$$

# Intuitition



# Existence of limits

We observe that the sequence of convex hulls $\{c(P^{(n)})\}_{n\in\mathbb{N}}$ form a nested sequence of compact sets.

## Proposition

$\lim c(P^{(n)}) = \bigcap c(P^{(n)})$

*Proof.* This is a direct consequence of the earlier proposition about the intersection of nested compact sets, satisfied because the $c(P^{(n)})$ are certainly nested and compact. ∎

# Existence of limits

## Proposition

$\lim c(F^{(n)})$ *exists, with respect to the Hausdorff metric.*

*Proof.* By the definition of fixed points, $F^{(n)} \subset F^{(n+1)}$. This implies the convex hulls $c(F^{(n)}) \subset c(F^{(n+1)})$ form a reverse-nested sequence of subsets. Lastly, $c(F^{(n)}) \subset c(P^{(0)})$ for all $n$. So $\{c(F^{(n)})\}$ satisfies the conditions of Proposition 3.9. ∎

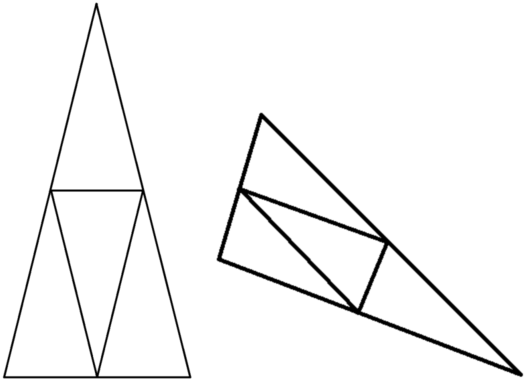
As a remark, this implies $\{c(F^{(n)})\}$ are a Cauchy sequence.

# Hausdorff distance

Recall the Hausdorff distance between two sets is given by

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(y,x)\right\}.$$

# Midpoint Triangle Property



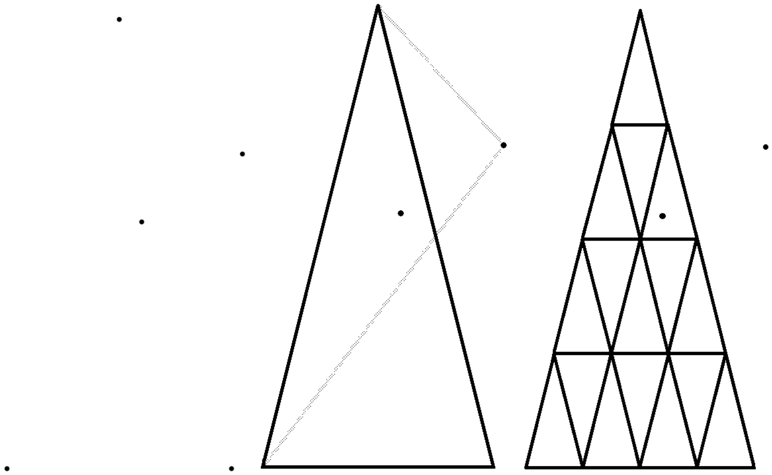
# Propositions

## Proposition

$\lim d_H(F^{(n)}, c(F^{(n)})) = 0$

*Proof.* Note $F^{(n)} \subset c(F^{(n)})$, so we consider the distance from $c(F^{(n)})$ to $F^{(n)}$. Let $x \in c(F^{(n)})$.
There are three noncollinear points in $F^{(n)}$ such that $x$ is in their convex hull. Taking $n$ iterations, we have $4^n$ congruent triangles, since these points are all fixed, and $x$ is in one of these $4^n$ triangles.

# Propositions



# Propositions

Let $T_n$ represent a triangle of the $n$th subdivision containing $x$. By the Schoenberg midpoint iteration, for arbitrary $\varepsilon > 0$, there exists an $N$ such that for $n > N$, the vertices $\{v_1, v_2, v_3\}$ of $T_n$ are within $\varepsilon$ distance of the centroid $c$ of $T_n$. Then

$$\inf_i d(x, v_i) \leq \sup_j d(c, v_j) < \varepsilon$$

This implies

$$d_H(F^{(n)}, c(F^{(n)})) = \sup_{x \in c(F^{(n)})} \inf_{y \in F^{(n)}} d(x, y) < \varepsilon. \blacksquare$$

# Propositions

## Proposition

$\lim d_H(P^{(n)}, F^{(n)}) = 0$

*Proof.* Note $F^{(n)} \subset P^{(n)}$, so the Hausdorff distance is equal to

$$d_H(P^{(n)}, F^{(n)}) = \sup_{x \in P^{(n)}} \inf_{y \in F^{(n)}} d(x, y).$$

# Propositions

Let $p' \in P^{(n+1)}$ arbitrarily. By definition of the GMI, there exists at least one pair $p, q \in P^{(n)}$ such that $\frac{p+q}{2} = p'$. Then by the Midpoint Triangle Property,

$$d(p', F^{(n+1)}) \leq \frac{1}{2} d(p, F^{(n)}) \leq \frac{1}{2} \sup_{x \in P^{(n)}} \inf_{y \in F^{(n)}} d(x, y).$$

# Propositions

Since $p'$ was arbitrary and $P^{(n)}$ and $F^{(n)}$ are finite sets, we may choose $p'$ such that

$$\sup_{z \in P^{(n+1)}} d(z, F^{(n+1)}) = d(p', F^{(n+1)}).$$

This is precisely the inequality

$$d_H(P^{(n+1)}, F^{(n+1)}) \leq \frac{1}{2} d_H(P^{(n)}, F^{(n)}). \blacksquare$$

# Propositions

## Proposition

$\lim d_H(P^{(n)}, c(F^{(n)})) = 0$

*Proof.* Triangle inequality:

$$d_H(P^{(n)}, c(F^{(n)})) \leq d_H(P^{(n)}, F^{(n)}) + d_H(F^{(n)}, c(F^{(n)}))$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

for appropriate $N$.

# Main result

## Theorem

*The sequence of convex hulls of fixed points $F^{(n)}$ and the sequence of convex hulls of $P^{(n)}$ have the same limit. That is,*

$$\lim c(F^{(n)}) = \lim c(P^{(n)}) = \bigcap c(P^{(n)})$$

*And consequently* $\lim d_H(P^{(n)}, c(P^{(n)})) = 0.$

# Main result

*Proof.* For notation, let $\mathcal{F} = \lim c(F^{(n)})$ and $\mathcal{P} = \lim c(P^{(n)})$. Seeking contradiction, suppose $\mathcal{F} \subsetneq \mathcal{P}$. Then $d_H(\mathcal{F}, \mathcal{P}) \neq 0$.

By proposition 5.2, the convergent sequence $\left\{c(F^{(n)})\right\}_{n \in \mathbb{N}}$ is also a Cauchy sequence. Therefore, for all $\varepsilon$ there exists $N$ such that for $n, m > N$, $d_H(c(F^{(n)}), c(F^{(m)})) < \varepsilon$.

# Main result

By compactness, choose $p \in c(P^{(N)})$ and $f \in c(F^{(N)})$ such that

$$d(p, f) = d_H(c(P^{(N)}), c(F^{(N)})).$$

Since $\mathcal{F} \subsetneq \mathcal{P}$, $d(p, f)$ is necessarily positive. Now, the midpoint of $f$ and $p$ is in $F^{(N+1)}$ by proposition 4.3. Denote $g = \frac{p+f}{2}$. Then

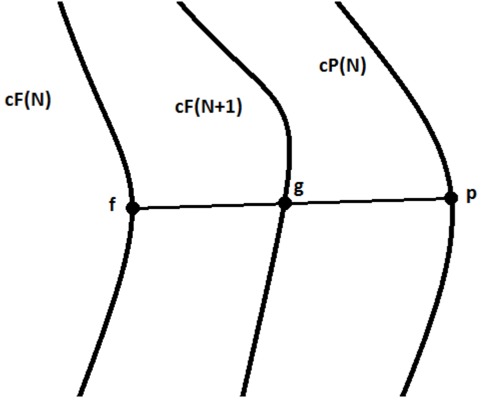
$$d_H(c(F^{(N)}), c(F^{(N+1)})) \geq \inf_{y \in c(F^{(N)})} d(g, y) \geq \frac{1}{2} d(p, f) > 0.$$

Passing to the limit,

$$\lim_N d_H(c(F^{(N)}), c(F^{(N+1)})) > 0.$$

contradicting the fact that $\left\{c(F^{(k)})\right\}$ is a Cauchy sequence. Conclude $\mathcal{F} = \mathcal{P}$ necessarily.

# Main result



# Main result

Consequently ,

$$d_H(P^{(n)}, c(P^{(n)})) \leq d_H(P^{(n)}, c(F^{(n)})) + d_H(c(F^{(n)}), c(P^{(n)}))$$
$$\lim d_H(P^{(n)}, c(P^{(n)})) = 0. \blacksquare$$

# Main result

The takeaway from this result is that the $P^{(n)}$ become dense in their convex hulls $c(P^{(n)})$, and, since the convex hulls $c(P^{(n)})$ converge to their intersection $\bigcap_{n \in \mathbb{N}} c(P^{(n)})$, the $P^{(n)}$ become dense in the limit.

The $c(F^{(n)})$ are approaching the limit from the inside, the $c(P^{(n)})$ are approaching the limit from the outside, and the $P^{(n)}$ are dense in the limit where they meet.

# Unit square example

We still don't know anything about the shape of the limit, whether it becomes smooth or elliptical. So we considered the simplest case of the GMI on a unit square.

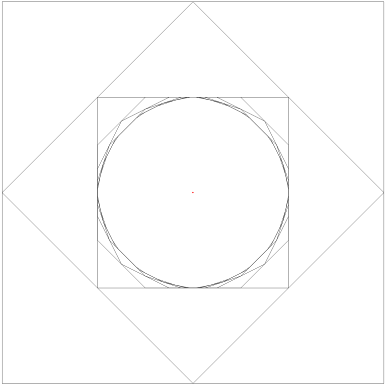
# Unit square example

It turns out that the unit square limits on the boundary to a 44-gon! (The inside will be filled up because of our major theorem on density.)

This is unexpected behavior because we expected a circle, or at least an ellipse, or at the very least something smooth. While this behavior makes the GMI unlikely to directly generalize to O'Rourke's conjecture, it is still very interesting.

# Unit square example

Surely if we ignore multiplicity and fixed points, the unit square would limit to a circle, right? It looks like it!



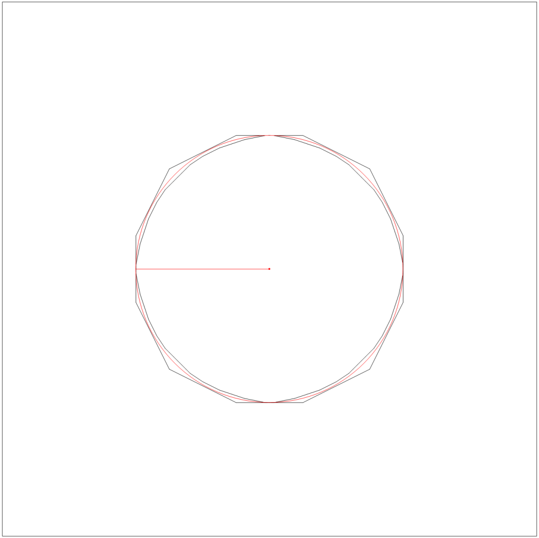
# Unit square example

But actually our calculations in Mathematica suggest that this limit is not a circle.

After five iterations, the area of the limit is less than the area of the smallest circle containing all the known limit points.

This suggests that O'Rourke's conjecture may be false. However, there are many differences between the GMI and the O'Rourke iteration, including using centroids instead of midpoints, using $\mathbb{R}^3$ instead of $\mathbb{R}^2$, and taking only centroids on the boundary instead of all centroids. In general, however, this sort of iteration probably does not produce strictly elliptical bodies.

## Unit Square Example



# What we showed

In the GMI, the points become dense in the limit, which is equivalently the union of the convex hulls of fixed points or the intersection of the convex hulls of all points.

This process turns a discrete set into a subset of $\mathbb{R}^2$.

# What we didn't show

We weren't able to prove O'Rourke's conjecture about the centroid-of-faces iteration converging to an ellipsoid.

In fact, our results do not include a characterization of the shape of the limit set.

But the theory we developed still may lead in some way to a solution of O'Rourke's conjecture.

# Multiplicity and fixed points

We developed the concept of multiplicity in order to find some structure in this ever-increasing set of points under the GMI.

However, we saw that fixed points from multiplicity lead to a strange convergence result on the unit square, which does not become smooth but converges to a 44-gon.

Since fixed points do not lead to the convergence results we expected, they may not be the right tool to work towards a proof of O'Rourke's conjecture (although they are interesting in their own right!). However, the proof techniques may still apply.

# Future work

- Prove a similar density result ($\lim d_H(P^{(n)}, c(P^{(n)}) = 0$) without counting multiplicity and without the fixed points that come from multiplicity.
- Prove results on the shape of the limit — does the GMI converge to an ellipse, some sort of generalized ellipse, or a union of ellipses? (Or just to some blob)
- Consider a generalized centroid iteration, taking centroids of three points instead of midpoints. More generally, we could iterate by any convex combination of points and hope for some sort of result.
- Extend to three dimensions or to general $\mathbb{R}^n$. Some of our proofs rely on the structure of the plane. (Or to general normed linear spaces — eg consider functions instead of points.)

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