Properties of Various Graphs Associated to a Group

There are many graph-theoretical constructs that can be associated to a group. Investigating which properties these graphs possess and how these properties relate to the algebraic properties of the group provides a broad range of opportunities for undergraduate research.

One of the most natural constructions is the subgroup lattice of a group. The vertices are subgroups of the group and there is an edge between two vertices if and only if one subgroup is contained in the other with no intermediate subgroup properly between them. The graph theoretical properties of these graphs have been extensively investigated in prior REU's and master's theses [7,8,11,15,18], but there are still some topics that could be investigated (e.g. graph pebbling).

The subgroup graph is also a lattice. Determining the Möbius function of that lattice is critical in determining the probability that k distinct elements generate a group. In her thesis [9], Christie Tosh Bowerman has addressed this problem for a number of groups, but there are many families of groups left to investigate.

Another graph associated to a group G is the non-commuting graph. This is the graph whose vertices are the elements of $G – Z(G)$ (namely the elements that do not commute with everything) with an edge between the vertex associated to x and that associated to y if and only if $xy \neq yx$. Some properties of this graph have been investigated for certain groups [1,4,10], interesting conjectures have been made [21], and these graphs have been the subject of a prior REU [13,14,17,19,20], but many topics remain (e.g. graph pebbling).

A generalization of the above is to fix an n and consider the graph obtained by first taking its vertices to be the elements of G and with an edge between x and y if and only if $(xy)^n \neq (yx)^n$ and then removing any isolated vertices. In [16], results are obtained for this graph when $n = 2$. There are more properties to investigate both for the case when $n = 2$ and when $n > 2$. One could also replace $(xy)^n \neq (yx)^n$ with other conditions of interest.

Another related graph is the non-cyclic graph which is obtained by connecting vertex x with vertex y if and only if the subgroup generated by x and y is not cyclic and then removing isolated vertices. This graph has been studied in [2,3], but again many questions remain to be answered.

An example where the vertices are not elements of G is the conjugacy graph, whose vertices are the conjugacy classes of non-central elements and two vertices are connected by an edge if their associated conjugacy classes have orders having a common divisor greater than 1 [6].

A final example is the G-graph of a group G and a set of generators $S = \{s_1, s_2, \ldots, s_k\}$. The vertices are left cosets of $\langle s_i \rangle$ and two vertices are connected by an edge if the associated cosets intersect non-trivially. The Hamiltonian and Eulerian properties and the planarity of these graphs have been investigated [5,12], but much remains to be done.

In the author's experience, the synergism between the group-theoretical and graph-theoretical aspects of these problems makes this project particularly appealing to undergraduates. Upon completion of the project, participants should have a greatly deepened understanding of both group theory and graph theory.

Prerequisites: Calculus and one semester of abstract algebra. Some exposure to graph theory is desirable, but not required.

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