Convergence and orchestrated divergence of Polygons

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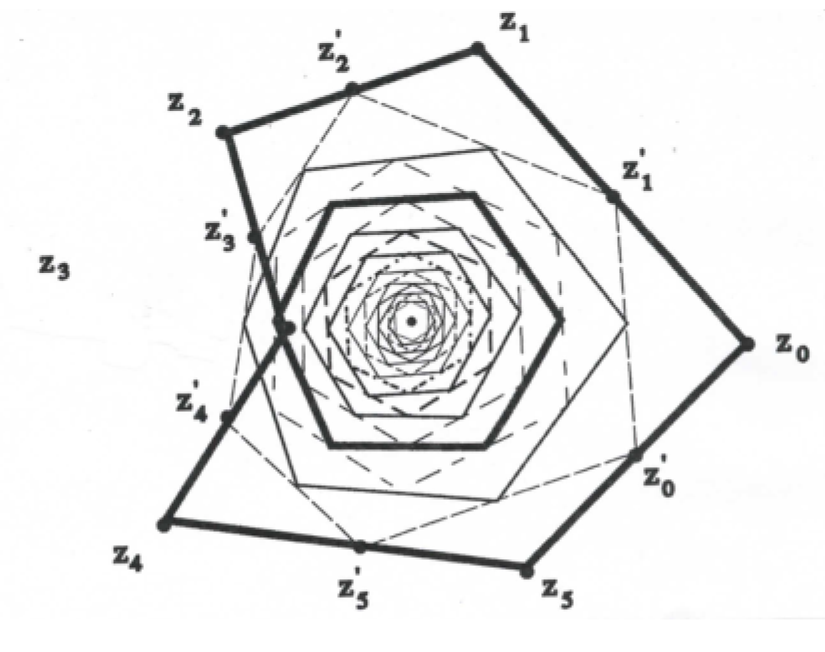
Let Π be a closed polygon in the plane with vertices $z_0, z_1, \ldots, z_{k-1}$. We arbitrarily pick a point from each of the sides $z_0z_1, z_1z_2, \ldots, z_{k-1}z_0$. Denote these points, respectively, by $z_0^{(1)}, z_1^{(1)}, \ldots, z_{k-1}^{(1)}$, which we use as the vertices of a new polygon $\Pi^{(1)}$. Apply the same procedure to derive polygon $\Pi^{(2)}$. Proceeding in an iterative fashion, we obtain polygon $\Pi^{(n)}$. The following diagram gives an illustration of the iteration process in which the midpoints of all sides of the current polygon are picked as the vertices of the next polygon.



The analysis group (2013 REU) has proved the following theorem.

Theorem 0.1. *Let $0 < c \leq 1/2$ be a given constant. Select $z_{i-1}^{(1)}$ on the edge $z_{i-1}z_i$, such that*

$$\min\left(dist\left(z_{i-1}^{(1)}, z_{i-1}\right), dist\left(z_{i-1}^{(1)}, z_i\right)\right) \geq c\; dist\left(z_{i-1}, z_i\right), \qquad i = 0, 1 \ldots, k-1.$$

The same requirement is enforced in the selection of the vertices of $\Pi^{(n)}$, $n \geq 2$. Then the sequences of polygons $\Pi^{(n)}$ converge to a point, as $n \to \infty$.

This generalizes the process known as "the midpoint iteration of polygons", initially dubbed as the Monthly Problem # 3547, proposed in 1932 by M. Rosenman and solved first in 1933 by R. Huston [3]. This problem has attracted the attention and interest of many mathematicians and scientists alike. Notably, in 1950, Schoenberg [4] introduced the concept of finite Fourier analysis to tackle this problem. Part of Schoenberg's work was summarized in the book "Fourier Analysis on Finite Groups and Applications" by Terras [5]. The finite Fourier analysis technique allows Schoenberg to establish an exponential rate at which the polygon $\Pi^{(n)}$ converges to the centroid of the original points $z_0, z_1, \ldots, z_{k-1}$. Oldenburger [2] phrased this problem in terms of matrix iterations (see also Treatman and Wickham [6]). Ding at al [1] considered this problem as a special case of Markov chains.

The 2013 REU analysis group has also studied situations in which the iterations of polygons diverge. Through computer simulations, they observed that the patterns of divergences can be orchestrated by the actions of dihedral groups.

The 2014 analysis group will work on the following problems.

1. In the above theorem, we will make the constant C vary with n, the iteration parameter. We denote the sequence by C_n. We will explore conditions on C_n so that the iteration of polygons converges to a point. A primary goal is to establish a convergence rate of polygons based on the rate at which the sequence C_n approaches zero.

2. We will use Monte Carlo method to simulate the process of iteration of polygons. We will utilize the computer graphic facilities via Matlab, and design some interesting convergent/divergent patterns.

3. We will investigate the convergence (in probability) of stochastic polygons. Let $\Pi^{(j)}$ be a closed polygon in the plane, and $z_0^{(j)}, z_1^{(j)}, \ldots, z_{k-1}^{(j)}$ be its vertices. Parametrize each side $z_i^{(j)} z_{i+1}^{(j)}$ by

$$t \mapsto (1-t)z_i^{(j)} + tz_{i+1}^{(j)} \quad t \in [0,1], \quad i = 0, 1, \ldots, k-1 \quad (z_k^{(j)} = z_0^{(j)}).$$

Let $g_i^{(j)}(t)$ denote the density function of a probabilistic distribution on the side $z_{i-1}^{(j)} z_{i+1}^{(j)}$. Choose the new vertex $z_i^{(j+1)}$ according to this distribution. That is, the probability of choosing $z_{i-1}^{(j+1)}$ with parameter t in the interval $[t_1, t_2]$, $(0 \le t_1 < t_2 \le 1)$ is given by

$$\int_{t_1}^{t_2} g_i^{(j)}(t)dt.$$

This gives rise to the new (stochastic) polygon $\Pi^{(j+1)}$. Assume that all the random variables $z_i^{(j)}$ $(j = 0, 1, \ldots,\ i = 0, 1, \ldots, k-1)$ are independent. Explore conditions under which the sequence of (stochastic) polygons $\Pi^{(n)}$ converges in probability to a single point x_0. That is, for any given $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\{\max_{0\le i<k} \operatorname{dist}(z_i^{(n)} - x_0) > \epsilon\} = 0.$$

We will start with the uniform distribution and fully use the random number generator function via Matlab. Theoretically, we will try to establish a Chebyshev type estimate. A similar problem concerning Bernstein polynomials based on scattered points has been studied by Wu, Sun, and Ma [7].

4. For the mathematically adventurous, we can explore the convergence/divergence of (hyper) space polygons. In space, the possibilities are limitless.

Prerequisite: A solid grasp of calculus and linear algebra concepts is required, and some familiarity with probability theory is desired.

References

1. J. Ding, L. Richard Hitt, and X. M. Zhang, Markov chains and dynamic geometry of polygons, Linear Algebra and Applications **367** (2003), 255-270.

2. R. Oldenburger, Infinite powers of matrices and characteristic roots, Duke Mathematical Journal **6** (1940), 357-361.

3. Martin Rosenman, Problem no. 3547, American Mathematical Monthly **39** (1932), P. 239. A solution of this problem was given by R. E. Huston in American Mathematical Monthly **40** (1933), 184-185.

4. I. J. Schoenberg, Mathematical Time Exposures, Mathematical Association of America, Washington, DC, 1982.

5. A. Terras, Fourier Analysis on Finite Groups and Applications, Cambridge University Press, 1999.

6. S. Treatman and C. Wickham, Constructible approximations of regular polygons, American Mathematical Monthly **107**(2000), 911-922.

7. Z. Wu, X. Sun, and L. Ma, Sampling scattered data with Bernstein polynomials: deterministic and stochastic error estimates, Advances in Computational Mathematics 38 (2013), no. 1, 187205.