Miscellaneous Topics in Group Theory, Combinatorics, and Diophantine Equations

- Let S_n denote the symmetric group on n elements. It is easy to show that there are exactly 10 elements $x \in S_4$ such that x^{10} is the identity. More generally, we ask for which k there are exactly k elements $x \in S_n$ such that x^k is the identity. For a given n, what is the smallest such $k > 1$? What is the largest such $k < n!$? What about other families of groups?

- Given a group G, the conjugacy class of g is $\{xgx^{-1} \mid x\ G\}$. If A and B are subsets of G, then $AB = \{ab \mid a \in A \text{ and } b \in B\}$. It is well known that the product of any two conjugacy classes is a disjoint union of conjugacy classes. We propose investigating how many classes can be in that disjoint union. For example, it is known that in S_n the product of two non-trivial conjugacy classes cannot be a single conjugacy class. We would start by looking at S_n, but other groups could also be investigated.

- Given strings of length n consisting of $1,2,\ldots,m$ (with repeats allowed), how many have no non-decreasing (consecutive) strings of length k?

- You are given a deck of cards labeled 1 though n and an arbitrary permutation (shuffling) of that deck. Perform the following sequence of moves. At each stage, move the card on the top of the deck to the position in the deck corresponding to its number. For example, if we started with 3241 (top is on the left and bottom on the right), we would subsequently get 2431, 4231, 2314, 3214, 2134, and 1234 at which point we cannot move. For a given n, what is the longest sequence of moves possible? Considering all $n!$ possible permutations, what is the average number of moves?

- In the (unfortunately) now defunct magazine *Quantum*, George Berzsenyi defined $G(n,k)$ to be $\max\{\mathrm{GCD}((x + 1)^n + k, x^n + k) \mid x \text{ is an integer}\}$. He computed $G(n,2)$ and asked if other values could be found. The author found explicit formulas for $G(n,3)$, $G(n,4)$, and $G(n,5)$. He believes that formulas for larger values of k can be found using similar techniques.

- If a point lies at distances a, b, and c from the vertices of an equilateral triangle with a side-length of d, then

  $$a^4 + b^4 + c^4 + d^4 = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.$$

  Finding integer solutions to this equation has been thoroughly investigated (although there may be more to do). We pose the analogous problem for non-equilateral triangles in two dimensions. We also ask what happens for regular tetrahedra in three dimensions (and for regular simplices in higher dimensions).