Convergence of Polygons and Polyhedrons (X. Sun)

In 1887, Darboux, the same French mathematician who made fundamental contribution to the theory of Riemann integration, studied the problem of midpoint iteration of polygons [1]. Simply put, he constructed a sequence of polygons in which the vertices of a descendant polygon are the midpoints of its parent polygon, and are connected by edges in the same order as those of its parent polygon. He showed that such a sequence of polygons converges to their common centroid.

In proving this result, Darboux utilized the powerful mathematical tool we know today as the finite Fourier transform. For a long time period, however, neither Darboux's result nor his method was widely known. The same problem was proposed in 1932 by Rosenman as the Monthly Problem # 3547 [5], and had been studied by several authors, including I. J. Schoenberg [6] who also employed the finite Fourier transform technique.

Hintikka (REU 2013) and the author (faculty mentor) generalized the problem of midpoint iteration of polygons [2]. Their scheme for the construction of a polygon sequence not only gives freedom in selecting the vertices of a descendant polygon but also allows the polygon generating procedure itself to vary from one step to another. They show under some mild restrictions that a sequence of polygons thus constructed converges to a single point, which can be predetermined under certain circumstances. Their main mathematical tools are ergodicity coefficient [4] and Parron's theory on positive matrices [3]. One can further generalize the problem in many practical and mathematically interesting ways. We propose two such generalizations in the following.

Problem I. Given n points from the unit circle, one takes the collection of all the midpoints of the arcs connecting these points pairwise (there are a total of $n(n-1)$ such arcs). Repeating the procedure, one gets the k-th collection of points obtained in this fashion. We can show that the point set becomes dense in the unit circle when k approaches infinity. Is the point set uniformly distributed in the unit circle under Weyl's criterion? If so, what is the rate of discrepancy?

Problem II. For a given tetrahedron, one uses the centroids of their faces to form a new tetrahedron. Repeating this procedure, one gets a sequence of tetrahedrons. Using a result from [2], we can show that the sequence of tetrahedrons thus formed converges to their common centroid. Now presented with a convex polyhedron, we triangulate its faces, and use the centroids of the triangles to build a new convex, polyhedron. Repeating in this fashion, one builds a sequence of polyhedrons. What is the asymptotic behavior of the sequence of polyhedrons? Many numerical simulations suggest that such a sequence of polyhedrons converges to an ellipsoid (a degenerated one under some circumstances).

Prerequisites: A solid background in calculus and linear algebra, and optimally, some experience with real analysis.

References

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[4] Ipsen, I. C. F. and Teresa M. Selee, Ergodicity Coefficients Defined by Vector Norms, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 153-200.

[5] M. Rosenman, Problem no. 3547, Amer. Math. Monthly, 39 (1932), pp. 239. A solution of this problem was given by R. E. Huston in Amer. Math. Monthly 40 (1933), 184-185.

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