Modeling and Stability Analysis of a Zika Virus Dynamics

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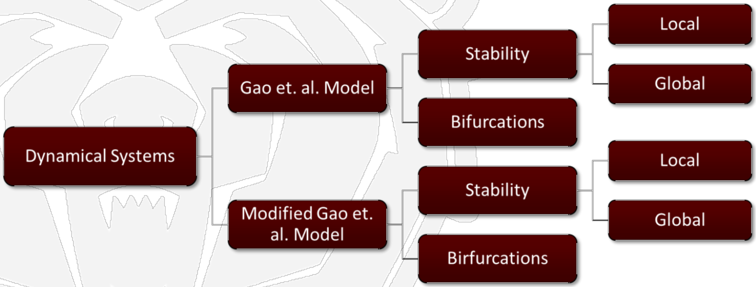
Outline


Dynamical Systems
Gao et. al. Model
Stability
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Modified Gao et. al. Model
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Dynamical Systems

- A dynamical system may be defined as

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, \lambda)$$
$$\mathbf{z} \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^m,$$
$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n.$$

- A system has a equilibrium point when there exists a $\mathbf{z_0} \in \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{z_0}, \lambda) = 0$

Equilibria Types and Stability Analysis

- Non-hyperbolic Equilibrium
  - Center Manifold Theorem
  - Liapunov Functions
- Hyperbolic Equilibrium
  - Linearization
  - Stable Manifold Theorem and Hartman-Grobman Theorem

Crash Course on Zika Virus

- Zika Virus (ZIKV) was first isolated in humans in 1954.
- The current ZIKV epidemic ongoing in the Americas began in Brazil in Apr. 2015.
- ZIKV is transmitted from mosquito bites.
- Evidence suggests it may also be transmitted sexually.
- 1 out of 5 infected people develop symptoms.
- Symptoms include mild fever, rash, conjunctivitis and joint pain.
- Evidence also suggest ZIKV may also increase the chance of microcephaly in newborns and cause Guillain-Barr syndrome.
- The current epidemic is expected to worsen due to Olympics and increasing mosquito population.

The Original Zika Model

The Gao et. al. model for ZIKV is represented by:

$$
\begin{aligned}
\dot{S}_h &= -ab\frac{I_v}{N_h}S_h - \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h \\
\dot{E}_h &= \theta\left(ab\frac{I_v}{N_h}S_h + \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h\right) - \nu_h E_h \\
\dot{I}_{h1} &= \nu_h E_h - \gamma_{h1} I_{h1} \\
\dot{I}_{h2} &= \gamma_{h1} I_{h1} - \gamma_{h2} I_{h2} \\
\dot{A}_h &= (1-\theta)\left(ab\frac{I_v}{N_h}S_h + \beta\frac{\kappa E_h + I_{h1} + _{h2}}{N_h}S_h\right) - \gamma_h A_h \\
\dot{R}_h &= \gamma_{h2} I_{h2} + \gamma_h A_h \\
\dot{S}_v &= \mu_v N_v - ac\frac{\eta E_h + I_{h1}}{N_h}S_v - \mu_v S_v \\
\dot{E}_v &= ac\frac{\eta E_h + I_{h1}}{N_h}S_v - (\nu_v + \mu_v)E_v \\
\dot{I}_v &= \nu_v E_v - \mu_v I_v
\end{aligned}
$$

Original Model: Equilibrium Points

- The original model has 7 Disease-Free equilibrium points of the form DFE $= (s_h, 0, 0, 0, 0, r_h, s_v, 0, 0)$.
- They are all non-hyperbolic.
- There are no Endemic equilibrium points (EE).

Local Center Manifold Theorem

If

- $\mathbf{f}(\mathbf{0}) = 0$ and $D\mathbf{f}(\mathbf{0})$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$, and
- $\mathbf{F}(\mathbf{0}) = \mathbf{G}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{F}(\mathbf{0}) = D\mathbf{G}(\mathbf{0}) = [0]$,

Then

- the system can be written in diagonal form $\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y})$ and $\dot{\mathbf{y}} = P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y})$,
- there exists an $\mathbf{h}$ that defines the local center manifold and satisfies $D\mathbf{h}(\mathbf{x})[C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x}))] - P\mathbf{h}(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = 0$,
- and the flow on the center manifold is defined by the system of differential equations $\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$

Original Model: Jacobian

$$J(\mathrm{DFE}) =$$

$$\left[\begin{array}{ccc|cccccc}
0 & 0 & 0 & 0 & 0 & -\frac{abs_h}{s_h+r_h} & -\frac{s_h\beta\kappa}{s_h+r_h} & -\frac{s_h\beta}{s_h+r_h} & -\frac{s_h\beta\tau}{s_h+r_h} \\
0 & 0 & 0 & \gamma_h & 0 & 0 & 0 & 0 & \gamma_{h2} \\
0 & 0 & 0 & 0 & \mu_v & \mu_v & -\frac{acs_v\eta}{s_h+r_h} & -\frac{acs_v}{s_h+r_h} & 0 \\
\hline
0 & 0 & 0 & -\gamma_h & 0 & \frac{abs_h(1-\theta)}{s_h+r_h} & \frac{s_h\beta(1-\theta)\kappa}{s_h+r_h} & \frac{s_h\beta(1-\theta)}{s_h+r_h} & \frac{s_h\beta(1-\theta)\tau}{s_h+r_h} \\
0 & 0 & 0 & 0 & \mu_v-\nu_v & 0 & \frac{acs_v\eta}{s_h+r_h} & \frac{acs_v}{s_h+r_h} & 0 \\
0 & 0 & 0 & 0 & \nu_v & -\mu_v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{abs_h\theta}{s_h+r_h} & \frac{s_h\beta\theta\kappa}{s_h+r_h}-\nu_h & \frac{s_h\beta\theta}{s_h+r_h} & \frac{s_h\beta\theta\tau}{s_h+r_h} \\
0 & 0 & 0 & 0 & 0 & 0 & \nu_h & -\gamma_{h1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{h1} & -\gamma_{h2}
\end{array}\right]$$

Diagonal Form

- Models can be written in diagonal form,

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) \quad \longrightarrow \quad \dot{\mathbf{z}} = \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix},$$

where C has c eigenvalues with zero real parts and P has s eigenvalues with negative real part, to yield

$$\begin{aligned} \dot{\mathbf{x}} &= C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y}) \\ \dot{\mathbf{y}} &= P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Original Model: $\mathbf{F}(\mathbf{x}, \mathbf{y})$

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} -ab\frac{y_9}{N_{xy}}(x_1 + s_h) - \beta\frac{\kappa y_2 + y_3 + \tau y_4}{N_{xy}}(x_1 + s_h) \\ \gamma_{h2} y_4 + \gamma_h y_5 \\ \mu_v(y_8 + y_9) - ac\frac{\eta y_2 + y_3}{N_{xy}}(x_7 + s_v) \end{bmatrix}$$

Original Model: $\mathbf{G}(\mathbf{x},\mathbf{y})$

$$\mathbf{G}(\mathbf{x},\mathbf{y}) =$$

$$\begin{bmatrix} (1-\theta)\left(ab\frac{y_9}{N_{xy}}(x_1+s_h)+\beta\frac{\kappa y_2+y_3+\tau y_4}{N_{xy}}(x_1+s_h)-ab\frac{y_9}{s_h+r_h}s_h-\beta\frac{\kappa y_2+y_3+\tau y_4}{s_h+r_h}s_h\right) \\ ac\frac{\eta y_2+y_3}{N_{xy}}(x_7+s_v)-ac\frac{\eta y_2+y_3}{s_h+r_h}s_v \\ 0 \\ \theta\left(ab\frac{y_9}{N_{xy}}(x_1+s_h)+\beta\frac{\kappa y_2+y_3+\tau y_4}{N_{xy}}(x_1+s_h)-ab\frac{y_9}{s_h+r_h}s_h-\beta\frac{\kappa y_2+y_3+\tau y_4}{s_h+r_h}s_h\right) \\ 0 \\ 0 \end{bmatrix}$$

Local Center Manifold Theorem

If

- ✓ $\mathbf{f}(\mathbf{0}) = 0$ and $D\mathbf{f}(\mathbf{0})$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$, and
- $\mathbf{F}(\mathbf{0}) = \mathbf{G}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{F}(\mathbf{0}) = D\mathbf{G}(\mathbf{0}) = [0]$,

Then

- ✓ the system can be written in diagonal form $\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y})$ and $\dot{\mathbf{y}} = P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y})$,
- there exists an $\mathbf{h}$ that defines the local center manifold and satisfies $D\mathbf{h}(\mathbf{x})[C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x}))] - P\mathbf{h}(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = 0$,
- and the flow on the center manifold is defined by the system of differential equations $\dot{\mathbf{x}} = C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$

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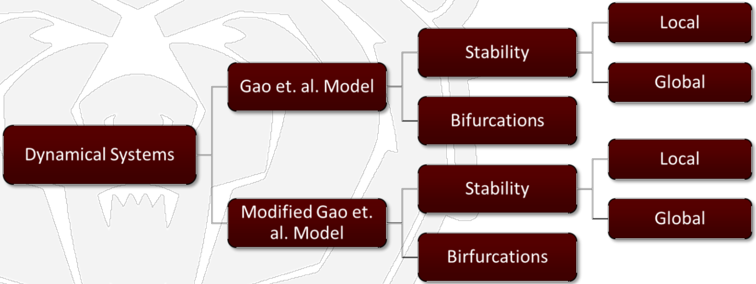
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Matrix Theoretic Method for Global Stability

- Epidemiology models can be written in a compartmentalized form

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) \quad \longrightarrow \quad \dot{\mathbf{z}} = \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix}$$

- The disease compartment is further split into $\mathcal{F}$, which represents the transmission terms, and $\mathcal{V}$, which is the transition terms

$$\begin{aligned} \dot{\mathbf{x}} &= \mathcal{F}(x, y) - \mathcal{V}(x, y) \qquad & x &\in \mathbb{R}^p, \\ \dot{\mathbf{y}} &= g(x, y) \qquad & y &\in \mathbb{R}^q, \quad n = p + q \end{aligned}$$

Matrix Theoretic Method for Global Stability

Then we split the disease compartment, $\dot{\mathbf{x}}$, into its linear and non-linear components.

$$\dot{\mathbf{x}} = (\mathbf{F} - \mathbf{V})\mathbf{x} - \mathbf{f}(x, y)$$

where

$$\mathbf{F}(x, y) = \left[\frac{\delta \mathcal{F}_i}{\delta x_j}(0, y_0) \right], \quad \mathbf{V}(x, y) = \left[\frac{\delta \mathcal{V}_i}{\delta x_j}(0, y_0) \right]$$

and

$$\mathbf{f}(x, y) = (\mathbf{F} - \mathbf{V})\mathbf{x} - \mathcal{F}(x, y) + \mathcal{V}(x, y)$$

Theorem 2.1 [Shuai and Van Den Driessche]

Goal is to construct a Liapunov function, Q.

If

- $\mathbf{f}(x, y) \geq 0$ in $\Gamma \subset \mathbb{R}_+^{p+q}$,
- $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, and
- $R_0 \leq 1$.

Then the function

$$Q = \omega^T \mathbf{V}^{-1} \mathbf{x}$$

where ω is the left eigenvector of $\mathbf{FV^{-1}}$ associated with R_0 is a Liapunov function for the system on Γ.

Liapunov Functions

If

- E is an open subset of $\mathbb{R}^n$ containing z_0,
- $\mathbf{f} \in C^1(E)$ and $\mathbf{f}(z_0) = 0$, and
- there exists $V \in C^1(E)$ satisfying $V(z_0) = 0$ and $V(z) > 0$ if $z \neq z_0$.

Then

(a) if $\dot{V}(z) \leq 0$ for all $z \in E$, z_0 is stable,

(b) if $\dot{V}(z) < 0$ for all $z \in E$, z_0 is asymptotically stable,

(c) if $\dot{V}(z) > 0$ for all $z \in E$, z_0 is unstable.

Basic Reproduction Number

- The reproduction number, R_0, is defined biologically as the average number of individuals infected by a single infected individual entering the susceptible population.
- R_0 is defined mathematically as the spectral radius, $\rho(A)$, of matrix A which is the Next Generation Matrix.

Theorem 2.2 [Shuai and Van Den Driessche]

If

- $\Gamma \subset \mathbb{R}_+^{p+q}$ is compact such that $(0, y_0) \in \Gamma$,
- Γ is positively invariant with respect to the system,
- $\dot{\mathbf{y}} = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is GAS in $\mathbb{R}_+^q$,
- $\mathbf{f}(x, y) \geq 0$ with $\mathbf{f}(x, y_0) = 0$ in Γ, and
- $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, $\mathbf{V}^{-1}\mathbf{F}$ be irreducible.

Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

Original Model: Feasible Region Γ

$$\dot{N}_h = 0 \qquad \dot{N}_v = 0$$
$$N_h = N_h(0) \qquad N_v = N_v(0)$$

Thus,

$$\Gamma = \{S_h, E_h, I_{h1}, I_{h2}, A_h, R_h, S_v, E_v, I_v \in \mathbb{R}^9_+ \mid S_h + E_h + I_{h1} + I_{h2} + A_h + R_h \leq N_h(0),\ S_v + E_v + I_v \leq N_v(0)\}$$

Original Model: Equilibrium Points in Γ

- The Disease-Free equilibrium, $(s_h, 0, 0, 0, 0, r_h, s_v, 0, 0) \in \Gamma$.
- The DFE is GAS in the disease-free subsystem because we let $N_h(0) = s_h + r_h$ and $N_v(0) = s_v$.

Theorem 2.2 [Shuai and Van Den Driessche]

If

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Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- × If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

Original Model: $\mathcal{F}$ and $\mathcal{V}$

$$\mathcal{F} = \begin{bmatrix} \theta\left(ab\frac{I_v}{N_h}S_h + \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h\right) \\ 0 \\ 0 \\ ac\frac{\eta E_h + I_{h1}}{N_h}S_v \\ 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} \nu_h E_h \\ \gamma_{h1} I_{h1} - \nu_h E_h \\ \gamma_{h2} I_{h2} - \gamma_{h1} I_{h1} \\ (\nu_v + \mu_v) E_v \\ \mu_v I_v - \nu_v E_v \end{bmatrix}$$

Original Model: F(x,y) and V(x,y)

$$\mathbf{F} = \begin{bmatrix} \frac{s_h \beta\theta\kappa}{s_h + r_h} & \frac{s_h\beta\theta}{s_h + r_h} & \frac{s_h\beta\theta\tau}{s_h + r_h} & 0 & \frac{s_h a b\theta}{s_h + r_h} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{ac\eta s_v}{s_h + r_h} & \frac{acs_v}{s_h + r_h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0$$

and

$$\mathbf{V} = \begin{bmatrix} \nu_h & 0 & 0 & 0 & 0 \\ -\nu_h & \gamma_{h1} & 0 & 0 & 0 \\ 0 & -\gamma_{h1} & \gamma_{h2} & 0 & 0 \\ 0 & 0 & 0 & \mu_v + \nu_v & 0 \\ 0 & 0 & 0 & -\nu_v & \mu_v \end{bmatrix}$$

Original Model: f(x,y)

$$\mathbf{f}(x,y) = (\mathbf{F} - \mathbf{V})\mathbf{x} - \mathcal{F}(x,y) + \mathcal{V}(x,y)$$

$$\mathbf{f} = \begin{bmatrix} \dfrac{\theta(N_h s_h - (s_h + r_h)S_h)(abI_v + \beta(\kappa E_h + I_{h1} + \tau I_{h2}))}{(s_h + r_h)N_h} \\ 0 \\ 0 \\ \dfrac{ac(N_h s_v - (s_h + r_h)S_v)(\eta E_h + I_{h1})}{(s_h + r_h)N_h} \\ 0 \end{bmatrix} \geq 0$$

$$\mathbf{f}(s_h, E_h, I_{h1}, I_{h2}, 0, r_h, s_v, E_v, I_v) \neq 0$$

Theorem 2.2 [Shuai and Van Den Driessche]

If

- ✓ $\Gamma \subset \mathbb{R}_+^{p+q}$ is compact such that $(0, y_0) \in \Gamma$,
- ✓ Γ is positively invariant with respect to the system,
- ✓ $\dot{\mathbf{y}} = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is GAS in $\mathbb{R}_+^q$,
- ✓ $\mathbf{f}(x, y) \geq 0$ with ~~$\mathbf{f}(x, y_0) = 0$~~ in Γ, and
- $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, ~~$\mathbf{V}^{-1}\mathbf{F}$ be irreducible~~.

Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- × If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

Original Model: $\mathrm{V}^{-1}(x, y)$

$$\mathbf{V}^{-1} = \begin{bmatrix} \nu_h^{-1} & 0 & 0 & 0 & 0 \\ \gamma_{h1}^{-1} & \gamma_{h1}^{-1} & 0 & 0 & 0 \\ \gamma_{h2}^{-1} & \gamma_{h2}^{-1} & \gamma_{h2}^{-1} & 0 & 0 \\ 0 & 0 & 0 & (\mu_v + \nu_v)^{-1} & 0 \\ 0 & 0 & 0 & \nu_v(\mu_v(\mu_v + \nu_v))^{-1} & \mu_v^{-1} \end{bmatrix} \geq 0$$

Original Model: Next Generation Matrix

$$\mathbf{FV}^{-1} =$$

$$\begin{bmatrix}
\frac{s_h \beta\theta(\kappa\gamma_{h1}\gamma_{h2} + \gamma_{h2}\nu_h + \gamma_{h1}\nu_h\tau)}{(s_h + r_h)\gamma_{h1}\gamma_{h2}\nu_h} & \frac{s_h\beta\theta(\kappa\gamma_{h2} + \gamma_{h1}\tau)}{(s_h + r_h)\gamma_{h1}\gamma_{h2}} & \frac{s_h\beta\theta\tau}{(s_h + r_h)\gamma_{h2}} & \frac{s_h ab\theta\nu_v}{(s_h + r_h)\mu_v(\mu_v + \nu_v)} & \frac{s_h ab\theta}{(s_h + r_h)\mu_v} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{acs_v(\eta\gamma_{h1} + \nu_h)}{(s_h + r_h)\gamma_{h1}\nu_h} & \frac{acs_v}{(s_h + r_h)\gamma_{h1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Theorem 2.2 [Shuai and Van Den Driessche]

If

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- ✓ Γ is positively invariant with respect to the system,
- ✓ $\dot{\mathbf{y}} = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is GAS in $\mathbb{R}_+^q$,
- ✓ $\mathbf{f}(x, y) \geq 0$ with ~~$\mathbf{f}(x, y_0) = 0$~~ in Γ, and
- ✓ $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, ~~$\mathbf{V}^{-1}\mathbf{F}$ be irreducible~~.

Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- × If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

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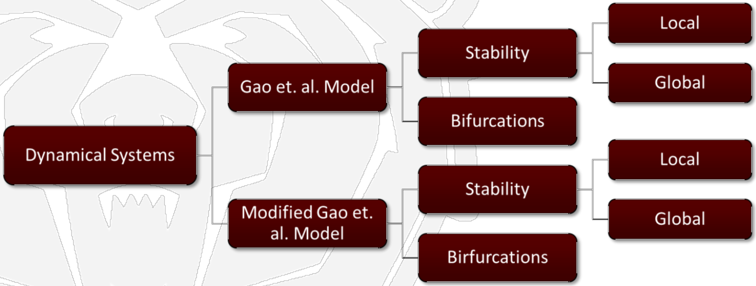
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The Modified Zika Model

The Gao et. al. model was modified according to Manore et. al.

$$\begin{aligned}
\dot{S}_h &= \mu_h(H_h - S_h) - ab\frac{I_v}{N_h}S_h - \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h \\
\dot{E}_h &= \theta\left(ab\frac{I_v}{N_h}S_h + \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h\right) - (\nu_h + \mu_h)E_h \\
\dot{I}_{h1} &= \nu_h E_h - (\gamma_{h1} + \mu_h)I_{h1} \\
\dot{I}_{h2} &= \gamma_{h1} I_{h1} - (\gamma_{h2} + \mu_h)I_{h2} \\
\dot{A}_h &= (1-\theta)\left(ab\frac{I_v}{N_h}S_h + \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h\right) - (\gamma_h + \mu_h)A_h \\
\dot{R}_h &= \gamma_{h2} I_{h2} + \gamma_h A_h - \mu_h R_h \\
\dot{S}_v &= \left(\Psi_v - \frac{r_v}{K_v}N_v\right)N_v - ac\frac{\eta E_h + I_{h1}}{N_h}S_v - \mu_v S_v \\
\dot{E}_v &= ac\frac{\eta E_h + I_{h1}}{N_h}S_v - (\nu_v + \mu_v)E_v \\
\dot{I}_v &= \nu_v E_v - \mu_v I_v
\end{aligned}$$

Modified Model: Equilibrium Points

- The modified model has one Disease-Free equilibrium point, DFE $= (H_h, 0, 0, 0, 0, 0, K_v, 0, 0)$.
- Use Routh-Hurwitz to ensure that they are hyperbolic and locally stable equilibrium points.
- We have numerically confirmed the existence of at least one Endemic equilibrium point (EE).

Linearization

- Study linear systems $\dot{\mathbf{z}} = A\mathbf{z}$.
- A system is linearized using the Jacobian

$$A = D\mathbf{f}(z_0) = \begin{bmatrix} \frac{\delta f_1}{\delta z_1}(z_0) & \dots & \frac{\delta f_1}{\delta z_n}(z_0) \\ \vdots & \ddots & \vdots \\ \frac{\delta f_n}{\delta z_1}(z_0) & \dots & \frac{\delta f_n}{\delta z_n}(z_0) \end{bmatrix}.$$

- Local stability of an equilibrium point of a linear system is determined by looking at the real part of the eigenvalues of $A = D\mathbf{f}(z_0)$.

Modified Model: Jacobian

$$
J(\mathrm{DFE}) =
\left[\begin{array}{cccc|ccccc}
-\mu_h & 0 & 0 & 0 & 0 & -ab & -\beta\kappa & -\beta & -\beta\tau \\
0 & -\mu_h & 0 & \gamma_h & 0 & 0 & 0 & 0 & \gamma_{h2} \\
0 & 0 & \mu_v - \Psi_v & 0 & 2\mu_v - \Psi_v & 2\mu_v - \Psi_v & -\frac{acK_v\eta}{H_h} & -\frac{acK_v}{H_h} & 0 \\
0 & 0 & 0 & -\gamma_h - \mu_h & 0 & ab(1-\theta) & \beta\kappa(1-\theta) & \beta(1-\theta) & \beta\tau(1-\theta) \\
\hline
0 & 0 & 0 & 0 & -\mu_v - \nu_v & 0 & \frac{acK_v\eta}{H_h} & \frac{acK_v}{H_h} & 0 \\
0 & 0 & 0 & 0 & \nu_v & -\mu_v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & ab\theta & \beta\theta\kappa - \mu_h - \nu_h & \beta\theta & \beta\theta\tau \\
0 & 0 & 0 & 0 & 0 & 0 & \nu_h & -\gamma_{h1} - \mu_h & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{h1} & -\gamma_{h2} - \mu_h
\end{array}\right]
$$

We have found the following sufficient conditions for DFE to be locally stable:

$$R + T + V + W + \mu_v > B\kappa$$

$$RT + RV + TV + RW + TW + VW + (R + T + V + W)\mu_v > B(\kappa(R + V + W + \mu_v) + \nu_h)$$

$$RTV + RTW + RVW + TVW + \mu_v(RT + RV + TV + RW + TW + VW)$$
$$> AS\eta\nu_v + B(\kappa(RV + RW + VW + \mu_v(R + V + W)) + \nu_h(R + V + \mu_v + \gamma_{h1}\tau))$$

$$RTVW + \mu_v(RTV + RTW + RVW + TVW)$$
$$> AS\nu_v(V\eta + W\eta + \nu_h) + B(\gamma_{h1}\mu_v + \kappa(RVW + \mu_v(RV + RW + VW)) + \nu_h(RV + R\mu_v + V\mu_v - R\gamma_{h1}\tau))$$

$$RTVW\mu_v > ASV\nu_v(W\eta + \nu_h) + B\mu_v(RVW\kappa + \nu_h(RV + R\gamma_{h1}\tau))$$

where

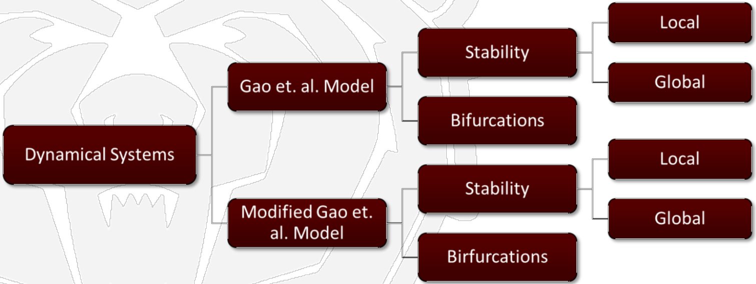
$$R = \mu_v + \nu_v \qquad S = ab\theta$$

$$W = \gamma_{h1} + \mu_h \qquad V = \gamma_{h2} + \mu_h$$

$$T = \mu_h + \nu_h \qquad B = \beta\theta$$

$$A = \frac{acK_v}{H_h}$$

Outline


Dynamical Systems
Gao et. al. Model
Modified Gao et. al. Model
Stability
Bifurcations
Stability
Birfurcations
Local
Global
Local
Global

Modified Model: Feasible Region Γ

$$\dot{N}_h = \mu_h H_h - \mu_h N_h$$

$$N_h(t) = H_h + k_3 e^{-\mu_h t}$$

$$\lim_{t\to\infty} N_h(t) = H_h$$

$$\dot{N}_v = r_v N_v \left(1 - \frac{N_v}{K_v}\right)$$

$$N_v(t) = K_v \left(1 + \frac{1}{e^{r_v t + k_4 Kv} - 1}\right)$$

$$\lim_{t\to\infty} N_v(t) = K_v$$

Thus, our feasible region is defined as:

$$\Gamma = \{S_h, E_h, I_{h1}, I_{h2}, A_h, R_h, S_v, E_v, I_v \in \mathbb{R}_+^9 \mid S_h + E_h + I_{h1} + I_{h2} + A_h + R_h \leq H_h,\ S_v + E_v + I_v \leq K_v\}$$

Modified Model: Equilibrium Points in Γ

- The Disease-Free equilibrium, DFE $= (H_h, 0, 0, 0, 0, 0, K_v, 0, 0) \in \Gamma$.
- The DFE $= (H_h, 0, 0, 0, 0, 0, K_v, 0, 0)$ is GAS in the disease-free subsystem because we have already seen that $\lim_{t\to\infty} N_h(t) = H_h$ and $\lim_{t\to\infty} N_v(t) = K_v$.

Theorem 2.2 [Shuai and Van Den Driessche]

If

- ✓ $\Gamma \subset \mathbb{R}_+^{p+q}$ is compact such that $(0, y_0) \in \Gamma$,
- ✓ Γ is positively invariant with respect to the system,
- ✓ $\dot{\mathbf{y}} = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is GAS in $\mathbb{R}_+^q$,
- $\mathbf{f}(x, y) \geq 0$ with ~~$\mathbf{f}(x, y_0) = 0$~~ in Γ, and
- $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, $\mathbf{V}^{-1}\mathbf{F}$ be irreducible.

Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- × If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

Modified Model: $\mathcal{F}$ and $\mathcal{V}$

$$\mathcal{F} = \begin{bmatrix} \theta\left(ab\frac{I_v}{N_h}S_h + \beta\frac{\kappa E_h + I_{h1} + \tau I_{h2}}{N_h}S_h\right) \\ 0 \\ 0 \\ ac\frac{\eta E_h + I_{h1}}{N_h}S_v \\ 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} (\nu_h + \mu_h)E_h \\ (\gamma_{h1} + \mu_h)I_{h1} - \nu_h E_h \\ (\gamma_{h2} + \mu_h)I_{h2} - \gamma_{h1}I_{h1} \\ (\nu_v + \mu_v)E_v \\ \mu_v I_v - \nu_v E_v \end{bmatrix}$$

Modified Model: F(x,y) and V(x,y)

$$\mathbf{F} = \begin{bmatrix} \beta\theta\kappa & \beta\theta & \beta\theta\tau & 0 & ab\theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{ac\eta K_v}{H_h} & \frac{acK_v}{H_h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0$$

and

$$\mathbf{V} = \begin{bmatrix} \mu_h + \nu_h & 0 & 0 & 0 & 0 \\ -\nu_h & \gamma_{h1} + \mu_h & 0 & 0 & 0 \\ 0 & -\gamma_{h1} & \gamma_{h2} + \mu_h & 0 & 0 \\ 0 & 0 & 0 & \mu_v + \nu_v & 0 \\ 0 & 0 & 0 & -\nu_v & \mu_v \end{bmatrix}$$

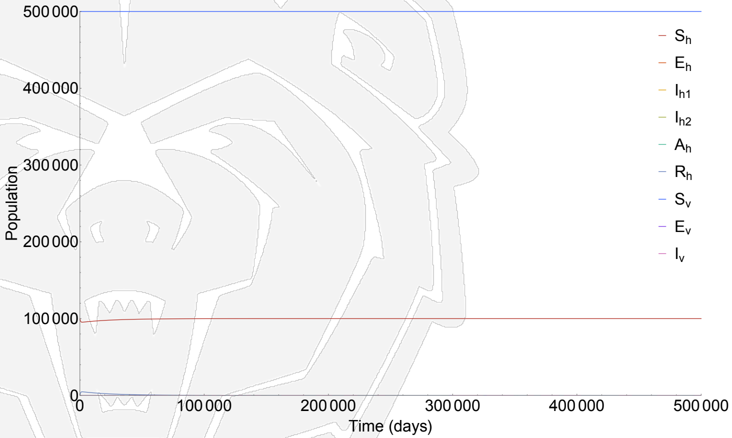
Modified Model: f(x,y)

$$\mathbf{f}(x,y) = (\mathbf{F} - \mathbf{V})\mathbf{x} - \mathcal{F}(x,y) + \mathcal{V}(x,y)$$

$$\mathbf{f} = \begin{bmatrix} \dfrac{\theta(N_h - S_h)(abI_v + \beta(\kappa E_h + I_{h1} + \tau I_{h2}))}{N_h} \\ 0 \\ 0 \\ \dfrac{ac(K_v N_h - H_h S_v)(\eta E_h + I_{h1})}{H_h N_h} \\ 0 \end{bmatrix} \geq 0$$

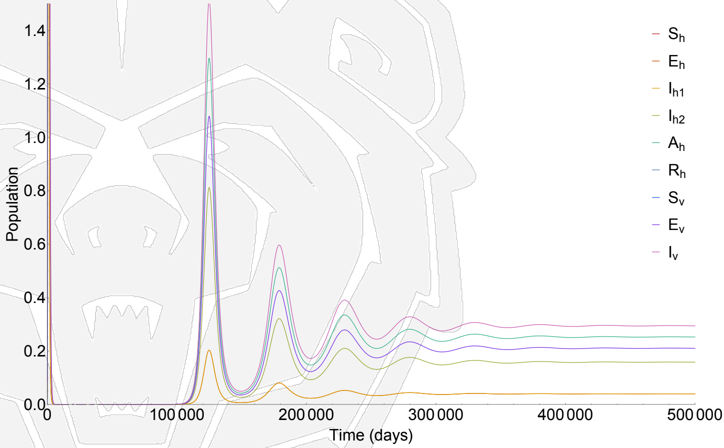
$$\mathbf{f}(H_h, E_h, I_{h1}, I_{h2}, 0, 0, K_v, E_v, I_v) \neq 0$$

Numerical Simulation: $R_0 < 1$



$$R_0 = 0.996127$$

Numerical Simulation: $R_0 > 1$



$$R_0 = 1.00605$$

Theorem 2.2 [Shuai and Van Den Driessche]

If

- ✓ $\Gamma \subset \mathbb{R}_+^{p+q}$ is compact such that $(0, y_0) \in \Gamma$,
- ✓ Γ is positively invariant with respect to the system,
- ✓ $\dot{\mathbf{y}} = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is GAS in $\mathbb{R}_+^q$,
- ✓ $\mathbf{f}(x, y) \geq 0$ with ~~$\mathbf{f}(x, y_0) = 0$~~ in Γ, and
- $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, ~~$\mathbf{V}^{-1}\mathbf{F}$ be irreducible~~.

Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- × If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

Modified Model: $V^{-1}(x, y)$

$$\mathbf{V}^{-1} = \begin{bmatrix} \frac{1}{\mu_h + \nu_h} & 0 & 0 & 0 \\ \frac{\nu_h}{(\gamma_{h1} + \mu_h)(\mu_h + \nu_h)} & \frac{1}{\gamma_{h1} + \mu_h} & 0 & 0 \\ \frac{\gamma_{h1}\nu_h}{(\gamma_{h1} + \mu_h)(\gamma_{h2} + \mu_h)(\mu_h + \nu_h)} & \frac{\gamma_{h1}}{(\gamma_{h1} + \mu_h)(\gamma_{h2} + \mu_h)} & \frac{1}{\gamma_{h2} + \mu_h} & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_v} \end{bmatrix} \geq 0$$

Modified Model: Next Generation Matrix

$$\mathbf{FV}^{-1} =$$

$$\begin{bmatrix} \frac{\beta\theta((\gamma_{h2}+\mu_h)(\kappa(\gamma_{h1}+\mu_h)+\nu_h)+\gamma_{h1}\nu_h\tau)}{(\gamma_{h1}+\mu_h)(\gamma_{h2}+\mu_h)(\mu_h+\nu_h)} & \frac{\beta\theta(\gamma_{h2}+\mu_h+\gamma_{h1}\tau)}{(\gamma_{h1}+\mu_h)(\gamma_{h2}+\mu_h)} & \frac{\beta\theta\tau}{\gamma_{h2}+\mu_h} & \frac{ab\nu_v\theta}{\mu_v(\mu_v+\nu_v)} & \frac{ab\theta}{\mu_v} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{acK_v(\eta(\gamma_{h1}+\mu_h)+\nu_h)}{H_h(\gamma_{h1}+\mu_h)(\mu_h+\nu_h)} & \frac{acK_v}{H_h(\gamma_{h1}+\mu_h)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 2.2 [Shuai and Van Den Driessche]

If

- ✓ $\Gamma \subset \mathbb{R}_+^{p+q}$ is compact such that $(0, y_0) \in \Gamma$,
- ✓ Γ is positively invariant with respect to the system,
- ✓ $\dot{\mathbf{y}} = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is GAS in $\mathbb{R}_+^q$,
- ✓ $\mathbf{f}(x, y) \geq 0$ with ~~$\mathbf{f}(x, y_0) = 0$~~ in Γ, and
- ✓ $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, ~~$\mathbf{V}^{-1}\mathbf{F}$ be irreducible~~.

Then

- If $R_0 < 1$, then the DFE is GAS in Γ.
- × If $R_0 > 1$, then the DFE is unstable, the system is uniformly persistent, and there exists at least one endemic equilibrium point.

Theorem 3.5 [Shuai and Van Den Driessche]

Goal is to construct a Liapunov function, $D(z)$.

If

- There exist functions $D_i : U \to \mathbb{R}$ and $G_{ij} : U \to \mathbb{R}$ and
- constants $a_{ij} \geq 0$ such that
- for every $1 \leq i \leq n$, $D_i' \leq \sum_{j=1}^{n} a_{ij} G_{ij}(z)$. And
- for $A = [a_{ij}]$, each directed cycle C of G has $\sum_{(s,r)\in\varepsilon(C)} G_{rs}(z) \leq 0$ where $\varepsilon(C)$ denotes the arc set of the directed cycle C.

Then the function

$$D(z) = \sum_{i=1}^{n} c_i D_i(z)$$

is a Liapunov function for the system.

Modified Model: D_i's

$$D_i = S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*} \quad \text{for } i = 1, 2, 3, 4$$

$$D_5 = E_h - E_h^* - E_h^* \ln \frac{E_h}{E_h^*}$$

$$D_6 = I_{h1} - I_{h1}^* - I_{h1}^* \ln \frac{I_{h1}}{I_{h1}^*}$$

$$D_7 = I_{h2} - I_{h2}^* - I_{h2}^* \ln \frac{I_{h2}}{I_{h2}^*}$$

$$D_j = S_v - S_v^* - S_v^* \ln \frac{S_v}{S_v^*} \quad \text{for } j = 8, 9$$

$$D_{10} = E_v - E_v^* - E_v^* \ln \frac{E_v}{E_v^*}$$

$$D_{11} = I_v - I_v^* - I_v^* \ln \frac{I_v}{I_v^*}$$

Modified Model: D_i's continued

Differentiation yields

$$
\begin{aligned}
D_i' \leq & \ ab\frac{I_v^* S_h^*}{N_h^*}\left(\frac{I_v}{I_v^*} - \ln\frac{I_v}{I_v^*} - \frac{I_v S_h}{I_v^* S_h^*} + \ln\frac{I_v S_h}{I_v^* S_h^*}\right) \\
& + \beta\kappa\frac{S_h^* E_h^*}{N_h^*}\left(\frac{E_h}{E_h^*} - \ln\frac{E_h}{E_h^*} - \frac{E_h S_h}{E_h^* S_h^*} + \ln\frac{E_h S_h}{E_h^* S_h^*}\right) \\
& + \beta\frac{S_h^* I_{h1}^*}{N_h^*}\left(\frac{I_{h1}}{I_{h1}^*} - \ln\frac{I_{h1}}{I_{h1}^*} - \frac{I_{h1} S_h}{I_{h1}^* S_h^*} + \ln\frac{I_{h1} S_h}{I_{h1}^* S_h^*}\right) \\
& + \beta\tau\frac{S_h^* I_{h2}^*}{N_h^*}\left(\frac{I_{h2}}{I_{h2}^*} - \ln\frac{I_{h2}}{I_{h2}^*} - \frac{I_{h2} S_h}{I_{h2}^* S_h^*} + \ln\frac{I_{h2} S_h}{I_{h2}^* S_h^*}\right) \\
:= & \ a_{4,11} G_{4,11} + a_{15} G_{15} + a_{36} G_{36} + a_{27} G_{27}
\end{aligned}
$$

Modified Model: D_i's continued

$$
\begin{aligned}
D_5' \leq\ & ab\theta \frac{I_v^* S_h^*}{N_h^*}\left(\frac{I_v S_h}{I_v^* S_h^*} - \ln \frac{I_v S_h}{I_v^* S_h^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*}\right) \\
& + \beta\theta\kappa \frac{S_h^* E_h^*}{N_h^*}\left(\frac{E_h S_h}{E_h^* S_h^*} - \ln \frac{E_h S_h}{E_h^* S_h^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*}\right) \\
& + \beta\theta \frac{S_h^* I_{h1}^*}{N_h^*}\left(\frac{I_{h1} S_h}{I_{h1}^* S_h^*} - \ln \frac{I_{h1} S_h}{I_{h1}^* S_h^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*}\right) \\
& + \beta\theta\tau \frac{S_h^* I_{h2}^*}{N_h^*}\left(\frac{I_{h2} S_h}{I_{h2}^* S_h^*} - \ln \frac{I_{h2} S_h}{I_{h2}^* S_h^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*}\right) \\
& := a_{54} G_{54} + a_{51} G_{51} + a_{53} G_{53} + a_{52} G_{52}
\end{aligned}
$$

$$
D_6' \leq \nu_h E_h^* \left(\frac{E_h}{E_h^*} - \ln \frac{E_h}{E_h^*} - \frac{I_{h1}}{I_{h1}^*} + \ln \frac{I_{h1}}{I_{h1}^*}\right) := a_{65} G_{65}
$$

Modified Model: D_i's continued

$$D_7' \leq \gamma_{h1} I_{h1}^* \left(\frac{I_{h1}}{I_{h1}^*} - \ln \frac{I_{h1}}{I_{h1}^*} - \frac{I_{h2}}{I_{h2}^*} + \ln \frac{I_{h2}}{I_{h2}^*} \right) := a_{76} G_{76}$$

$$D_j' \leq ac\eta \frac{E_h^* S_v^*}{N_h^*} \left(\frac{E_h}{E_h^*} - \ln \frac{E_h}{E_h^*} - \frac{E_h S_v}{E_h^* S_v^*} + \ln \frac{E_h S_v}{E_h^* S_v^*} \right)$$
$$+ ac \frac{I_{h1}^* S_v^*}{N_h^*} \left(\frac{I_{h1}}{I_{h1}^*} - \ln \frac{I_{h1}}{I_{h1}^*} - \frac{I_{h1} S_v}{I_{h1}^* S_v^*} + \ln \frac{I_{h1} S_v}{I_{h1}^* S_v^*} \right)$$
$$:= a_{95} G_{95} + a_{86} G_{86}$$

Modified Model: D_i's continued

$$D'_{10} \leq ac\eta \frac{E_h^* S_v^*}{N_h^*} \left(\frac{E_h S_v}{E_h^* S_v^*} - \ln \frac{E_h S_v}{E_h^* S_v^*} - \frac{E_v}{E_v^*} + \ln \frac{E_v}{E_v^*} \right)$$

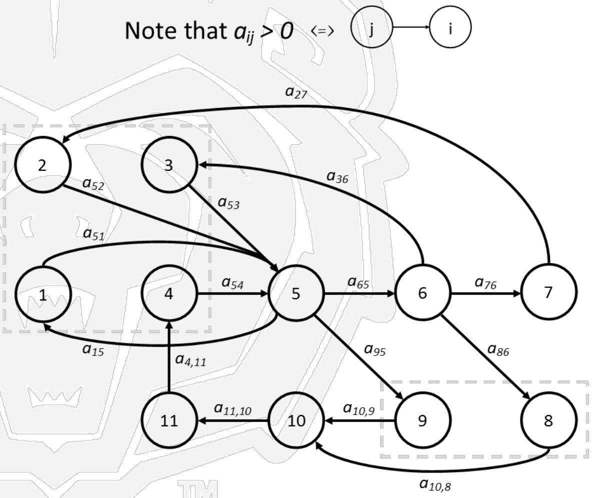
$$+ ac\eta \frac{I_{h1}^* S_v^*}{N_h^*} \left(\frac{I_{h1} S_v}{I_{h1}^* S_v^*} - \ln \frac{I_{h1} S_v}{I_{h1}^* S_v^*} - \frac{E_v}{E_v^*} + \ln \frac{E_v}{E_v^*} \right)$$

$$:= a_{10,9} G_{10,9} + a_{10,8} G_{10,8}$$

$$D'_{11} \leq \nu_v E_v^* \left(\frac{E_v}{E_v^*} - \ln \frac{E_v}{E_v^*} - \frac{I_v}{I_v^*} + \ln \frac{I_v}{I_v^*} \right) := a_{11,10} G_{11,10}$$

Modified Model: The Sydney Graph

Note that $a_{ij} > 0$ <=>



Modified Model: Directed Cycles

Consider:

1. the cycle beginning at node 1, going to node 5, and returning to node 1,
2. the cycle beginning at node 5, going to nodes 6 then 3, and returning to node 5,
3. the cycle beginning at node 5, going to nodes 6, 7, then 2, and returning to node 5,
4. the cycle beginning at node 5, going to nodes 9, 10, 11, then 4, and returning to node 5,
5. the cycle beginning at node 5, going to nodes 6, 8, 10, 11, then 4, and returning to node 5.

Theorem 3.5 [Shuai and Van Den Driessche]

If

- ✓ There exist functions $D_i : U \to \mathbb{R}$ and $G_{ij} : U \to \mathbb{R}$ and
- ✓ constants $a_{ij} \geq 0$ such that
- ✓ for every $1 \leq i \leq n$, $D_i' \leq \sum_{j=1}^{n} a_{ij} G_{ij}(z)$. And
- ✓ for $A = [a_{ij}]$, each directed cycle C of G has $\sum_{(s,r)\in\varepsilon(C)} G_{rs}(z) \leq 0$ where $\varepsilon(C)$ denotes the arc set of the directed cycle C.

Then the function

$$D(z) = \sum_{i=1}^{n} c_i D_i(z)$$

is a Liapunov function for the system.

Modified Model: Constructing Constants c_i

$$c_i = \frac{a_{51}}{a_{15}} = \frac{a_{52}}{a_{27}} = \frac{a_{53}}{a_{36}} = \frac{a_{54}}{a_{4,11}} = \theta$$

$$c_5 = 1$$

$$c_6 = \frac{a_{53}+a_{52}}{a_{65}} + \frac{a_{54}a_{10,8}}{a_{65}(a_{10,8}+a_{10,9})} = \frac{S_h^*\theta(abI_v^* I_{h1}^* + \beta(\eta E_h^* + I_{h1}^*)(I_{h1}^* + \tau I_{h2}^*))}{\nu_h E_h^* N_h^* (\eta E_h^* + I_{h1}^*)}$$

$$c_7 = \frac{a_{52}}{a_{76}} = \frac{\beta\theta\tau I_{h2}^* S_h^*}{\gamma_{h1} I_{h1}^* N_h^*}$$

$$c_j = \frac{a_{54}a_{10,8}}{a_{86}(a_{10,8}+a_{10,9})} = \frac{a_{54}a_{10,9}}{a_{95}(a_{10,8}+a_{10,9})} = \frac{b\theta I_v^* S_h^*}{cS_v^*(\eta E_h^* + I_{h1}^*)}$$

$$c_{10} = \frac{a_{54}}{a_{10,8}+a_{10,9}} = \frac{b\theta I_v^* S_h^*}{cS_v^*(\eta E_h^* + I_{h1}^*)}$$

$$c_{11} = \frac{a_{54}}{a_{11,10}} = \frac{ab\theta I_v^* S_h^*}{\nu_v E_v^* N_h^*}.$$

Modified Model: Constructing the Liapunov Function D

$$
\begin{aligned}
D &= c_i D_i + c_5 D_5 + c_6 D_6 + c_7 D_8 + c_j D_j + c_{10} D_{10} + c_{11} D_{11} \\
&= c_i \left(S_h - S_h^* - S_h^* \ln \frac{S_h}{S_h^*} \right) + c_5 \left(E_h - E_h^* - E_h^* \ln \frac{E_h}{E_h^*} \right) \\
&\quad + c_6 \left(I_{h1} - I_{h1}^* - I_{h1}^* \ln \frac{I_{h1}}{I_{h1}^*} \right) + c_7 \left(I_{h2} - I_{h2}^* - I_{h2}^* \ln \frac{I_{h2}}{I_{h2}^*} \right) \\
&\quad + c_j \left(S_v - S_v^* - S_v^* \ln \frac{S_v}{S_v^*} \right) + c_{10} \left(E_v - E_v^* - E_v^* \ln \frac{E_v}{E_v^*} \right) \\
&\quad + c_{11} \left(I_v - I_v^* - I_v^* \ln \frac{I_v}{I_v^*} \right).
\end{aligned}
$$

LaSalle's Invariance Principle

If

- ✓ $\Gamma \subset D \subset R^n$ be a compact positively invariant set,
- ✓ $V : D \to R$ be a continuously differentiable function,
- ✓ $\dot{V}(x(t)) \leq 0$ in Γ,
- $E \subset \Gamma$ be the set of all points in Γ where $\dot{V}(x) = 0$, and
- $M \subset E$ be the largest invariant set in E.

Then every solution starting in Γ approaches M as $t \to \infty$, that is,

$$\lim_{t\to\infty} \left(\underbrace{\inf_{z\in M} \|x(t) - z\|}_{\text{dist }(x(t), M)} \right) = 0.$$

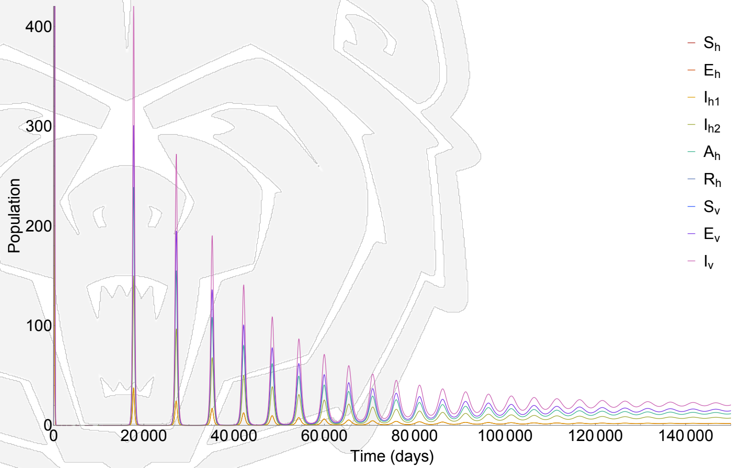
Modified Model: LaSalle's Invariance Principle

- Note that

$$\dot{D} = c_i \left(\frac{S_h - S_h^*}{S_h} \right) \dot{S}_h + c_5 \left(\frac{E_h - E_h^*}{E_h} \right) \dot{E}_h + c_6 \left(\frac{I_{h1} - I_{h1}^*}{I_{h1}} \right) \dot{I}_{h1} + c_7 \left(\frac{I_{h2} - I_{h2}^*}{I_{h2}} \right) \dot{I}_{h2}$$
$$+ c_j \left(\frac{S_v - S_v^*}{S_v} \right) \dot{S}_v + c_{10} \left(\frac{E_v - E_v^*}{E_v} \right) \dot{E}_v + c_{11} \left(\frac{I_v - I_v^*}{I_v} \right) \dot{I}_v$$

  which only equals zero at the EE.
- Since D is a Liapunov function and the EE is the only set where $\dot{D} = 0$, we can conclude that the EE is the largest invariant set $M \subset E$.
- Thus, under numerical simulations supporting Theorem 2.2 and LaSalle's Invariance Principle, we can conclude that the EE is GAS in Γ when $R_0 > 1$.

Numerical Simulation



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Stable Manifold Theorem

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system. Suppose that $f(0) = 0$ and that $Df(0) = 0$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$.

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

and there exists an $n-k$ dimensional differentiable manifold U tangent to the unstable subspace E^u of the corresponding linear system at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

Hartman-Grobman Theorem

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system. Suppose that $f(0) = 0$ and that the matrix $A = DF(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for $x_0 \in U$, there is an open interval $I_0 \in \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$

$$H \circ \phi_t(x_0) = e^{At} H(x_0)$$

i.e., H maps trajectories of the nonlinear system near the origin onto trajectories of the linear system near the origin and preserves the parametrization of time.

Local Center Manifold Theorem

Let $\mathbf{f} \in C^r(E)$ where E is an open subset of $\mathbb{R}^n$ containing the origin and $r \geq 1$. Suppose that $\mathbf{f}(0) = 0$ and that $D\mathbf{f}(0)$ has c eigenvalues with zero real parts and s eigenvalues with negative real parts, where $c + s = n$. The nonlinear system $x' = \mathbf{f}(x)$ can then be written in diagonal form

$$\begin{aligned} \dot{\mathbf{x}} &= C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{y}) \\ \dot{\mathbf{y}} &= P\mathbf{y} + \mathbf{G}(\mathbf{x}, \mathbf{y}). \end{aligned}$$

where $(x, y) \in \mathbb{R}^c \times \mathbb{R}^s$, $\mathbf{C}$ is a square matrix with c eigenvalues having zero real part, $\mathbf{P}$ is a square matrix with s eigenvalues having negative real part, and $\mathbf{F}(0) = \mathbf{G}(0) = 0$, $D\mathbf{F}(0) = D\mathbf{G}(0) = 0$; furthermore, there exists a $\delta > 0$ and function $h \in C^r(N_\delta(0))$ that defines the local center manifold and satisfies

Local Center Manifold Theorem

$$D\mathbf{h}(\mathbf{x})[C\mathbf{x} + \mathbf{F}(\mathbf{x}, \mathbf{h}(\mathbf{x}))] - P\mathbf{h}(\mathbf{x}) - \mathbf{G}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = 0$$

for $|x| < \delta$; and the flow on the center manifold W^c is defined by the system of differential equations $x' = \mathbf{C}x + \mathbf{F}(x, h(x))$ for all $x \in \mathbb{R}^c$ with $|x| < \delta$.

Routh-Hurwitz Criterion

Consider a three dimensional system. If the characteristic polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

satisfies the following:

$$a_1 > 0,$$
$$a_3 > 0,$$
$$a_1 a_2 - a_3 > 0$$

then the equilibrium point is locally stable.

Spectral Radius

The spectral radius of a matrix or bounded linear operator is the supremum among the absolute values of the elements in its spectrum. The spectrum of such a matrix is its set of eingenvalues, but the spectrum of a linear operator $T : V \to V$ is the set of scalars λ such that $T - \lambda I$ is non-invertible.

Partitioned Inversion (Using Shur Complements)

Given a partitioned matrix **M**:

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} (\mathbf{M}/\mathbf{D})^{-1} & -(\mathbf{M}/\mathbf{D})^{-1}\mathbf{B}\mathbf{A}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{M}/\mathbf{D})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{M}/\mathbf{D})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}$$

Where

$$(\mathbf{M}/\mathbf{D})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{M}/\mathbf{A})^{-1}\mathbf{C}\mathbf{A}^{-1}$$

if **A** and **D** are non-singular