Conjugacy Class Graphs of Dihedral and Permutation Groups

Genevieve Maalouf & Taylor Walker

Missouri State REU

Summer 2017

August 2, 2017
Overview

1. Preliminaries
2. Dihedral Groups
3. Permutation Groups
What’s a Group?

Definition

A group is a set $\mathcal{G}$ paired with a binary operation, $\ast$, that satisfies:

1. **Closure:** If $a, b \in \mathcal{G}$ then $a \ast b \in \mathcal{G}$.
2. **Associativity:** $a, b, c \in \mathcal{G}$, $(a \ast b) \ast c = a \ast (b \ast c)$
3. **Identity:** $\exists e \in \mathcal{G}$ such that $e \ast g = g \ast e = g$, $\forall g \in \mathcal{G}$.
4. **Inverses:** $\exists g^{-1} \in \mathcal{G}$ such that $g \ast g^{-1} = g^{-1} \ast g = e$, $\forall g \in \mathcal{G}$.

If the operation is commutative, the group is a commutative, or abelian, group. Examples include $(\mathbb{Z}, +)$ integers under addition, $D_{2n}$ the rotations and flips of an $n$-gon, and $S_n$ the set of all permutations of $n$ elements.
Conjugacy Classes

**Definition**

Let $G$ be a group. Then, the conjugacy class of $a \in G$ is the set denoted as $cl(a)$, where $cl(a) = \{xax^{-1} \mid x \in G\}$.

If $G$ is abelian, then each conjugacy class has size 1 because $xax^{-1} = xx^{-1}a = a$. 
Conjugacy Classes of $S_n$

- Each permutation can be represented as disjoint cycles. Example: $(13)(12)(46)(45) = (123)(456)$
- The conjugacy class of a $k$ cycle is the class containing all possible $k$ cycles, and only $k$ cycles. In other words, the cycle structure of every element in a conjugacy class is the same.
What's a Graph?

**Definition**
A graph, $G$, consists of a set of vertices, $V$, and a set of edges, $E$, which are defined by two vertices.

**Definition**
A complete graph, $K_n$, is a simple undirected graph in which every pair of $n$ distinct vertices is connected by a unique edge.
Let $G$ be a group. We create $\Gamma(G)$ by computing the conjugacy classes of $G - Z(G)$. A node is produced by every conjugacy class and labeled with the cardinality of the class, $c_i$. Lastly, an edge connects two vertices only if the $\gcd(c_{i_1}, c_{i_2}) > 1$. The main focus is to classify all graphs of $\Gamma(D_{2n} \times D_{2m})$ and to study the completeness of $\Gamma(S_n)$. 
Let
\[ a_{1,1}, a_{1,2} \ldots a_{1,k_1} \] be the conjugacy classes of \( G_1 \),
\[ a_{2,1}, a_{2,2} \ldots a_{2,k_2} \] be the conjugacy classes of \( G_2 \),
\[ \vdots \]
\[ a_{m,1}, a_{m,2}a_{1,k_m} \] be the conjugacy classes of \( G_m \)

The size of the conjugacy classes of \( G_1 \times G_2 \times \ldots G_m \) are all products in the form of
\[ a_{1,j_1} \cdot a_{2,j_2} \cdots a_{m,j_m} \] where
\[ 1 \leq j_1 \leq k_1, \quad 1 \leq j_2 \leq k_2, \ldots 1 \leq j_m \leq k_m. \]
Definition

An induced subgraph of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset.

Lemma

Let $G = \Gamma(G_1 \times G_2 \times \cdots \times G_k)$ be a graph. Then $\Gamma(G_i)$ is an induced subgraph of $G$ for $1 \leq i \leq k$.

Proof.

This is a consequence of how the size of the conjugacy class of $G_1 \times G_2 \times \cdots \times G_k$ are computed. Since we multiply every conjugacy class size by the size of $|\text{cl}(\text{id})|$, there is an exact copy of $\Gamma(G_i)$ for $1 \leq i \leq k$. Thus $\Gamma(G_i)$ is a subgraph $G$. □
To compute $\Gamma(S_3 \times S_3)$, since $S_3$ has conjugacy class sizes 1, 2, and 3 so since $\text{gcd}(2, 3) = 1$ $S_3$ isn’t complete, and neither is $\Gamma(S_3 \times S_3)$.

The size of each conjugacy class of $S_5$ are:
1, 10, 15, 20, 20, 24 and 30 and
the size of each conjugacy class of $S_7$ are:
1,, 21, 70, 105, 105, 210, 210, 280, 420, 420, 504, 504,
630, 720, and 840.
Since $\text{gcd}(10, 21) = 1$ $\Gamma(S_7 \times S_5)$ is not complete.
General Theorems

Theorem

If at least one of the graphs $\Gamma(G_1), \Gamma(G_2) \ldots \Gamma(G_k)$ are connected, then $\Gamma(G_1 \times G_2 \cdots \times G_k)$ is connected.
General Theorems

**Theorem**

Let $\mathcal{G} = \Gamma(G_1 \times G_2 \cdots \times G_k)$ be a complete graph, then $\Gamma(G_1), \Gamma(G_2), \ldots, \Gamma(G_k)$ are complete.

Again, the converse is not true. Let’s look at $\Gamma(S_5)$ and $\Gamma(S_7)$, both of which are complete. We show that $\Gamma(S_5 \times S_7)$ is not complete because $|\text{cl}((12))|$ of $S_5$ is 10 and $|\text{cl}((12))|$ of $S_7$ is 21.
Graphs of Dihedral Groups
In general, the three possibilities for the graph of $D_{2n}$ are:

$$
\Gamma(D_{2n}) = \begin{cases} 
    K_{n-1} / 2 \sqcup K_1 & n \equiv 1 \text{mod} 2 \\
    K_n / 2 - 1 \sqcup K_2 & n \equiv 2 \text{mod} 4 \\
    K_n / 2 + 1 & n \equiv 0 \text{mod} 4
\end{cases}
$$
The join of a $K_a$ and a $K_b$ along a $K_c$ will be denoted as $K_{\{a,b;c\}}$
The join of a $K_a$ and $K_b$ along a $K_d$, $K_a$ and $K_c$ along a $K_e$, $K_b$ and $K_c$ along a $K_f$, and the $K_d$, $K_e$, and $K_f$ along a $K_g$ will be denoted $K_{\{a,b,c;d,e,f;g\}}$. 
This would be a $K\{5,4,3;2,2,1;1\}$
Classification of $\Gamma(D_{2n} \times D_{2m})$

1. The cases where $n \equiv 1 \mod 2$ and $m \equiv 2 \mod 4$

   \[ K\left\{\frac{mn+4m+6n-11}{4}, n+3, \frac{m+6}{2}; n-1, \frac{m-2}{3}, 2; 0\right\}\]

   when $\gcd\left(\frac{m}{2}, n\right) = 1$

   \[ K\left\{\frac{mn+4m+6n-11}{4}, \frac{m+8}{2}, \frac{m+2n-4}{2}\right\}\]

   when $\gcd\left(\frac{m}{2}, n\right) > 1$

2. The cases where $n \equiv 0 \mod 4$ and $m \equiv 1 \mod 2$

   \[ K\left\{\frac{mn+6m+3n+2}{4}, n+6, \frac{n+2}{2}\right\}\]

   when $\gcd(m, \frac{n}{2}) = 1$

   \[ K\left\{\frac{mn+6m+3n+2}{4}, \frac{2m+n+8}{2}, \frac{2m+n+4}{2}\right\}\]

   when $\gcd(m, \frac{n}{2}) > 1$

3. The cases where $n \equiv 0 \mod 4$ and $m \equiv 2 \mod 4$

   \[ K\left\{\frac{mn+6m+6n+4}{4}, n+6; n+2\right\}\]

   when $\gcd\left(\frac{m}{2}, \frac{n}{2}\right) = 1$

   \[ K\left\{\frac{mn+6m+6n+4}{4}, m+n+8; m+n+4\right\}\]

   when $\gcd\left(\frac{m}{2}, \frac{n}{2}\right) > 1$
$\Gamma(D_{10} \times D_{12})$

$D_{10} \times D_{12}$ where $n \equiv 1 \mod 2$, $m \equiv 2 \mod 4$.

In this case, $n = 5$ and $m = 6$. Using the generalizations for $\Gamma(D_{2n})$, we know $\Gamma(D_{10})$ produces $K_2 \biguplus \ast$ and $\Gamma(D_{12})$ produces $K_2 \biguplus K_2$. We can obtain the graphs of this type with two subcases: $\gcd(n, \frac{m}{2}) = 1, \gcd(n, \frac{m}{2}) > 1$. When the $\gcd(n, \frac{m}{2}) > 1$, the graph can be generated using

$$K \{ (\frac{n-1}{2} + 2)(\frac{m}{2} + 3) - 8, (\frac{m}{2} + 3) + 2(\frac{n-1}{2} + 2) - 2; (\frac{m}{2} - 1) + 2(\frac{n-1}{2}) \}$$
However, when the \( \gcd(n, \frac{m}{2})=1 \), the graph is obtained using
Since the \( \gcd(5, \frac{6}{2}) = 1 \), we use the second formula to show \( \Gamma(D_{10} \times D_{12}) \) will form
Classification of $\Gamma(D_{2n} \times D_{2m})$

1. The cases where $n, m \equiv 1 \mod 2$
   - $K\left\{ \frac{mn+3m+3n-8}{4}, \frac{m+3}{2}, \frac{n+3}{2}; \frac{m-1}{2}, \frac{n-1}{2}, 1; 0 \right\}$
     when $\gcd(m, n) = 1$
   - $K\left\{ \frac{mn+3m+3n-8}{4}, \frac{m+n+4}{2}; \frac{m+n-2}{2} \right\}$
     when $\gcd(m, n) > 1$

2. The case where $n, m \equiv 0 \mod 4$
   - $K\frac{mn+6m+6n+20}{4}$

3. The case where $n, m \equiv 2 \mod 4$
   - $K\left\{ \frac{mn+6m+6n-12}{4}, m+6, n+6; m-2, n-2, 4; 0 \right\}$
     when $\gcd(\frac{m}{2}, \frac{n}{2}) = 1$
   - $K\left\{ \frac{mn+6m+6n-12}{4}, m+n+8; m+n-4 \right\}$
     when $\gcd(\frac{m}{2}, \frac{n}{2}) > 1$
\[ \Gamma(D_{10} \times D_{14}) \]

\[ D_{10} \times D_{14} \text{ where } n, m \equiv 1 \mod 2 \]

In this case \( n = 5 \) and \( m = 7 \). Since \( n \equiv 1 \mod 2 \), we use

\[ K_{\frac{n-1}{2}} \uplus \ast \]

to produce their graphs individually. Using the general formula, \( \Gamma(D_{10}) \) produces \( K_2 \uplus \ast \) and \( \Gamma(D_{14}) \) produces \( K_3 \uplus \ast \). We generalize how to produce the graph of their graph with two subcases: \( \gcd(n, m) > 1 \), \( \gcd(n, m) = 1 \). When \( \gcd(n, m) > 1 \), we generalize the graph of their product to be

\[ K\{(\frac{n-1}{2} + 2)(\frac{m-1}{2} + 2) - 4, (\frac{n-1}{2} + 2)(\frac{m-1}{2} + 2) - 1; (\frac{m-1}{2}) + (\frac{n-1}{2})\} \]
When the \( \gcd(n, m) = 1 \), we generalize the graph of their product to be
In $D_{10} \times D_{14}$, the gcd $(5, 7) = 1$, therefore we use the second equation to obtain our graph.

The $\Gamma(D_{10} \times D_{14})$ will produce...
Graphs of Permutation Groups
Definition

An integer partition of $x$ is a way of writing $x$ as a sum of positive integers. We denote the number of partitions there are of $x$ by $n(x)$. 
Theorem

For every prime $p \geq 5$, $S_p$ generates a complete graph.

To prove this, we need the following lemma:

Lemma

The smallest, nontrivial conjugacy class graphs of $S_n$ is the class of a single transposition, for $n \geq 7$.

For $n < 7$ the smallest class is still the class of transpositions, but there are other classes just as small.
The size of each conjugacy class is: 1, 10, 15, 20, 20, 24 and 30.
The cardinality of each conjugacy class in $S_p$ can be expressed as:

$$\frac{p!}{\prod_{i=1}^{p} (n_i)! \cdot i^{n_i}}$$

which is how we were able to show all the conjugacy classes had a factor in common, except for the conjugacy class of $p$-cycles. We then used the lemma to show that the smallest conjugacy classes (the transpositions) were

$$\binom{p}{2} = \frac{p(p-1)}{2} > p.$$
There is a direct correlation to the cycle structures of $S_p$ and the partitions of $n$, there are exactly $n(p)$ many conjugacy classes of $S_p$, where $n(x)$ is the number of partitions of a positive integer $x$. Thus, since the conjugacy class of the identity does not receive a vertex, $\Gamma(S_p) = K_{n(p)} - 1$. 
Now we study when $\Gamma(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k})$ generates a complete graph. In an earlier example, we noted that $\Gamma(S_5 \times S_7)$ is not complete. However, both $\Gamma(S_5)$ and $\Gamma(S_6)$ are complete, and $\Gamma(S_5 \times S_6)$ is as well. In fact, $\Gamma(S_5 \times S_n)$ iff $\Gamma(S_6 \times S_n)$. 
As mentioned earlier, the size of each conjugacy class of $S_5$ are:
1, 10, 15, 20, 20, 24 and 30
The size of each conjugacy class of $S_6$ are:
1, 15, 15, 40, 45, 90, 90, 120, 120, 144
The class of $S_5$ with 10 elements corresponds to the primes 2 and 5. Likewise, the class with 15 elements corresponds to the primes 3 and 5, and the class with 24 elements corresponds to the primes 2 and 3.

For $S_6$, the class with 40 elements corresponds to the primes 2 and 5. So if the class of 10 elements in $S_5$ is connected to a vertex, $v$, so is the class with 40 elements. Similarly the class with 45 elements is going to connect to every vertex the class of 15 elements (in $S_5$) connects to. And the class with 144 elements is going to connect to all the vertices the class of 24 elements connects to.
The graph $\Gamma(S_{p^2} \times S_p)$ is complete for all prime, $p \geq 5$. 
First, we showed that the only class, $\text{cl}(a)$ that’s size didn’t have a factor of $p$, of $S_p^2$, is the class of $p$ many $p$ cycles. For example, the only class of $S_9$ whose size is not divisible by 3 is $\text{cl}((123)(456)(789))$.

Next we showed that no conjugacy class had a size relatively prime to $|\text{cl}(a)|$. We already know that no class of $S_p$ has size exactly $p$, and we used a very similar argument to prove the same for $S_p^2$, using the transposition lemma.
Found general theorems about the completeness and connectedness of products of graphs.

Classified the conjugacy class graphs of the form of $\Gamma(D_{2n} \times D_{2m})$.

Found some conditions on when $\Gamma(S_k \times S_l)$ is complete.
Future Work

- Further the classification of when $\Gamma(S_n \times S_m)$ is complete.
- When is $\Gamma(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k})$ complete?
- What are the possibilities of $\Gamma(D_{2m} \times S_n)$?
The End