The Number of Conjugacy Classes Contained In Products of Conjugacy Classes

Herionexy Mounier-Acabá, Les Reid

Missouri State University

8/04/2017

What's a Group?

Definition

A group is a set G paired with a binary operation, $*$, that satisfies:

1. Closure: If $a, b \in G \implies a * b \in G$.
2. Associativity: $a, b, c \in G, (a * b) * c = a * (b * c)$
3. Identity: $\exists e \in G$ such that $e * g = g * e = g, \forall g \in G$.
4. Inverses: $\exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$, $\forall g \in G$

Example of a Group: S_n

- The set of all permutations of the set $\{1, \ldots, n\}$ is a group under function composition. This set is denoted as S_n.
- It is known that each permutation in S_n can be represented as a cycle or as a disjoint product of cycles. Examples:
  - In S_3, the permutation $\alpha = (123)$ is the function which:
    - $1 \mapsto 2$
    - $2 \mapsto 3$
    - $3 \mapsto 1$
  - In S_5, the permutation $\beta = (12)(345)$ is the function which:
    - $1 \mapsto 2$
    - $2 \mapsto 1$
    - $3 \mapsto 4$
    - $4 \mapsto 5$
    - $5 \mapsto 3$

Conjugacy Classes

Definition

Let G be a group. Then, the conjugacy class of $a \in G$ is the set denoted as $cl(a)$, where $cl(a) = \{xax^{-1} \mid x \in G\}$.

Example: In S_3, let's calculate $cl((12))$.

$(1)(12)(1)^{-1} = (12)$

$(12)(12)(12)^{-1} = (12)$

$(13)(12)(13)^{-1} = (23)$

$(23)(12)(23)^{-1} = (13)$

$(123)(12)(123)^{-1} = (123)(12)(132) = (23)$

$(132)(12)(132)^{-1} = (132)(12)(123) = (13)$

Therefore, $cl((12)) = \{(12), (13), (23)\}$

Conjugacy Classes in S_n

Theorem

The conjugacy class of any arbitrary permutation$\alpha \in S_n$ is the set of all the permutations such that has the same cycle type.

$\Delta(cl(a), cl(b))$

Definition

Let G be a group and $a, b \in G$. Then, $\Delta(cl(a), cl(b))$ denotes the number on conjugacy classes contained in $cl(a)cl(b) = \{xy \mid x \in cl(a), y \in cl(b)\}$

Basic Properties

Lemma

Let G, H and K be groups, where $a, b \in G$, $c, d \in H$ and $x \in Z(G)$ are arbitrary elements and $\phi : G \to K$ is an isomorphism. Then:

- $\Delta_G(cl(a), cl(b)) = \Delta_G(cl(b), cl(a))$
- $\Delta_{G\times H}(cl((a, c)), cl((b, d))) = \Delta_G(cl(a), cl(b)) \cdot \Delta_H(cl(c), cl(d))$
- $\Delta_G(cl(x), cl(a)) = 1$
- $\Delta_G(cl(xa), cl(b)) = \Delta_G(cl(a), cl(b))$
- $\Delta_G(cl(a), cl(b)) = \Delta_K(cl(\phi(a)), cl(\phi(b)))$

Result for Abelian Groups

Theorem

Let G be a group. Then, G is Abelian if and only if $\Delta(cl(a), cl(b)) = 1$ for all $a, b \in G$

Semi-Direct Products

Definition

Let H and K be groups, and $\phi : K \to Aut(H)$ be a homomorphism. Then the semi-direct product of H and K with respect to ϕ is the set denoted as $H \rtimes_\phi K$, where this set is the Cartesian product of H and K with the operation $\cdot$, where $(h_1, k_1) \cdot (h_2, k_2) = (h_1\phi(k_1)(h_2), k_1 k_2)$ for all $(h_1, k_1), (h_2, k_2) \in H \rtimes_\phi K$

Results in D_{2n}

Theorem

Let $n \geq 2$ be odd. Then, in $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$:

- $\Delta(cl(r^i), cl(r^j)) = 2$, for all $1 \leq i, j \leq n-1$
- $\Delta(cl(r^i), cl(s)) = 1$
- $\Delta(cl(s), cl(s)) = \dfrac{n+1}{2}$

Results in D_{2n}

Example: In $D_{10} = \langle r, s \mid r^5 = s^2 = 1, srs^{-1} = r^{-1} \rangle$, the conjugacy classes are the following:

- $cl(1) = \{1\}$
- $cl(r) = \{r, r^4\}$
- $cl(r^2) = \{r^2, r^3\}$
- $cl(s) = \{s, sr, sr^2, sr^3, sr^4\}$

Results in D_{2n}

Then:

- $cl(r)cl(r^2) = cl(r) \cup cl(r^2) = \{r, r^2, r^3, r^4\}$
- $cl(r)cl(s) = cl(s)$
- $cl(r^2)cl(s) = cl(s)$
- $cl(s)cl(s) = cl(1) \cup cl(r) \cup cl(r^2)$

Results in D_{2n}

Theorem

For any integer $n \geq 4$ such that $n \equiv 0 \pmod 4$, the following values of Δ holds for the conjugacy classes in D_{2n}.

- $\Delta(cl(r^i), cl(r^j)) = 2$ for all $i, j \in \{1, \ldots, \frac{n-2}{2}\}$
- $\Delta(cl(r^i), cl(s)) = \Delta(cl(r^i), cl(sr)) = 1$ for all $i \in \{1, \ldots, \frac{n}{2}\}$
- $\Delta(cl(s), cl(s)) = \Delta(cl(sr), cl(sr)) = 1 + \dfrac{n}{4}$
- $\Delta(cl(s), cl(sr)) = \dfrac{n}{4}$

Results in D_{2n}

Theorem

Let $n \geq 3$ be a positive integer. If $n \equiv 2 \ (mod\ 4)$, then the following values of Δ hold true for D_{2n}

- $\Delta(cl(r^i), cl(r^j)) = 2$ for all $i, j \in \{1, \ldots, \frac{n-2}{2}\}$
- $\Delta(cl(r^i), cl(s)) = \Delta(cl(r^i), cl(rs)) = 1$
- $\Delta(cl(s), cl(s)) = \Delta(cl(rs), cl(s)) = \Delta(cl(rs), cl(rs)) = \dfrac{n+2}{4}$

Results in $Q_{2^{n+1}}$

Theorem

The group with the set denoted as $Q_{2^{n+1}}$ and group presentation $\langle h, x \mid h^{2^n} = x^4 = 1, xhx^{-1} = h^{-1}, h^{2^{n-1}} = x^2\rangle$ is the generalized quaternion group of order 2^{n+1}. Then:

- $\Delta(cl(h^i), cl(h^j)) = 2$ for all $i, j \in \{1, \ldots, 2^{n-1} - 1\}$
- $\Delta(cl(h^i), cl(x)) = \Delta(cl(h^i), cl(x^3)) = 1$ for all $i \in \{1, \ldots, 2^n - 1\}$
- $\Delta(cl(x), cl(x^3)) = 1 + 2^{n-1}$

Results in QD_{2^n}

Theorem

The group denoted as QD_{2^n} is the quasidihedral group of order 2^n, which has the following group presentation:
$\langle r, s \mid r^{2^{n-1}} = s^2 = 1, srs^{-1} = r^{2^{n-2}-1} \rangle$. Moreover, it has the following values for Δ:

- $\Delta(cl(r^i), cl(r^j)) = 2$
- $\Delta(cl(r^i), cl(s)) = \Delta(cl(r^i), cl(rs)) = 1$
- $\Delta(cl(s), cl(s)) = \Delta(cl(rs), cl(rs)) = 1 + 2^{n-3}$
- $\Delta(cl(s), cl(rs)) = 2^{n-3}$

Results In Semi-Direct Products

Theorem

Let $p \equiv 1 \ (mod\ 3)$. Then
$C_p \rtimes C_3 = \langle a, b \mid a^p = b^3 = 1, bab^{-1} = a^g \rangle$, where g has order 3 in $(\mathbb{Z}_p)^\times$, and also:

- $\Delta(cl(a^i), cl(a^j)) = 3$, for all $i\langle g \rangle \neq j\langle g \rangle$
- $\Delta(cl(a^i), cl(a^i)) = 2$
- $\Delta(cl(a^i), cl(b)) = \Delta(cl(a^i), cl(b^2)) = 1$
- $\Delta(cl(b), cl(b^2)) = \dfrac{p+2}{3}$

Results In Semi-Direct Products

Theorem

Let $p, q \in \mathbb{Z}^+$ be any two odd primes. Suppose that $q \mid p-1$. Then $C_p \rtimes C_q = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^g \rangle$, where g has order q in $(\mathbb{Z}_p)^\times$, and also:

- $\Delta(cl(a^i), cl(a^j)) \leq q$, for all $i\langle g \rangle \neq j\langle g \rangle$
- $\Delta(cl(a^i), cl(a^i)) \leq \dfrac{q+1}{2}$
- $\Delta(cl(a^i), cl(b^u)) = 1$ for all $u \in \{1, \ldots, p-1\}$
- $\Delta(cl(b^i), cl(b^{-i})) = 1 + \dfrac{p-1}{q}$
- $\Delta(cl(b^s), cl(b^t)) = 1$ for all $s \not\equiv -t \pmod p$

Example: $C_{13} \rtimes C_3$

1	1	1	1	1	1	1
1	1	5	1	1	1	1
1	5	1	1	1	1	1
1	1	1	2	3	3	3
1	1	1	3	2	3	3
1	1	1	3	3	2	3
1	1	1	3	3	3	2

Example: $C_{19} \rtimes C_3$

1	1	1	1	1	1	1	1	1
1	1	7	1	1	1	1	1	1
1	7	1	1	1	1	1	1	1
1	1	1	2	3	3	3	3	3
1	1	1	3	2	3	3	3	3
1	1	1	3	3	2	3	3	3
1	1	1	3	3	3	2	3	3
1	1	1	3	3	3	3	2	3
1	1	1	3	3	3	3	3	2

Example: $C_{11} \rtimes C_5$

1	1	1	1	1	1	1
1	1	1	1	3	1	1
1	1	1	3	1	1	1
1	1	3	1	1	1	1
1	3	1	1	1	1	1
1	1	1	1	1	2	3
1	1	1	1	1	3	2

Example: $C_{31} \rtimes C_5$

1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	7	1	1	1	1	1	1
1	1	1	7	1	1	1	1	1	1	1
1	1	7	1	1	1	1	1	1	1	1
1	7	1	1	1	1	1	1	1	1	1
1	1	1	1	1	3	4	4	4	4	5
1	1	1	1	1	4	3	4	5	4	4
1	1	1	1	1	4	4	3	4	5	4
1	1	1	1	1	4	5	4	3	4	4
1	1	1	1	1	4	4	5	4	3	4
1	1	1	1	1	5	4	4	4	4	3

Example: $C_{41} \rtimes C_5$

1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	9	1	1	1	1	1	1	1	1
1	1	1	9	1	1	1	1	1	1	1	1	1
1	1	9	1	1	1	1	1	1	1	1	1	1
1	9	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	3	5	4	5	5	4	4	4
1	1	1	1	1	5	3	4	5	5	4	4	4
1	1	1	1	1	4	4	3	4	4	5	5	5
1	1	1	1	1	5	5	4	3	5	4	4	4
1	1	1	1	1	5	5	4	5	3	4	4	4
1	1	1	1	1	4	4	5	4	4	3	5	5
1	1	1	1	1	4	4	5	4	4	5	3	5
1	1	1	1	1	4	4	5	4	4	5	5	3

Example: $C_{29} \rtimes C_7$

1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	5	1	1	1	1
1	1	1	1	1	5	1	1	1	1	1
1	1	1	1	5	1	1	1	1	1	1
1	1	1	5	1	1	1	1	1	1	1
1	1	5	1	1	1	1	1	1	1	1
1	5	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	3	4	5	4
1	1	1	1	1	1	1	4	3	4	5
1	1	1	1	1	1	1	5	4	3	4
1	1	1	1	1	1	1	4	5	4	3

Results In Semi-Direct Products

Theorem

Let $n = {p_1}^{k_1} {p_2}^{k_2} \dots {p_l}^{k_l}$. If $3 \mid p_i - 1$, and g is an element of order 3 in $(\mathbb{Z}_{{p_i}^{k_i}})^\times$ for all $1 \leq i \leq l$, then $C_n \rtimes C_3$, with the presentation $\langle a, b \mid a^n = b^3 = 1, bab^{-1} = a^g \rangle$ has the following values for Δ:

- $\Delta(cl(a^j), cl(a^j)) = 2$, for all $1 \leq j \leq n$
- $\Delta(cl(a^u), cl(a^v)) = 2$ or 3, for all $u \neq vg^\alpha$
- $\Delta(cl(a^m), cl(b)) = \Delta(cl(a^m), cl(b^2)) = 1$

Results In Semi-Direct Products (Heisenberg group over C_p)

Theorem

Let p be an odd prime. Then the Heisenberg group over C_p has the group presentation $\langle a, b, x \mid a^p = b^p = x^p = 1, ab = ba, xax^{-1} = ab, xbx^{-1} = b \rangle$ and the values for Δ are either 1 or p.

Results In Semi-Direct Products (Modular Group)

Theorem

Let p be an odd prime. Then the Modular group has order p^3, has the group presentation $\langle a, b \mid a^{p^2} = b^p = 1, bab^{-1} = a^{1+p} \rangle$, and the values for Δ are either 1 or p.

Results In Semi-Direct Products

Theorem

Let p and q be odd primes such that $q \mid p-1$. Then $(C_p \times C_p) \rtimes C_q$ has the group presentation $\langle a, b, x \mid a^p = b^p = x^q = 1, ab = ba, xax^{-1} = a^{g^{k_1}}, xbx^{-1} = b^{g^{k_2}} \rangle$, where g is an element of order q in $(\mathbb{Z}_p)^\times$ and $q \nmid k_1, k_2$. Also, this group has the following values of Δ:

- $\Delta(cl(a^i b^u), cl(a^j b^v)) \leq q$ for all $i\langle g \rangle \neq j\langle g \rangle$ and $u\langle g \rangle \neq v\langle g \rangle$
- $\Delta(cl(a^i b^t), cl(a^i b^t)) \leq \dfrac{q+1}{2}$
- $\Delta(cl(a^i), cl(a^i b^j)) = \Delta(cl(b^j), cl(a^i b^j)) = \Delta(cl(a^i), cl(b^j)) = q$
- $\Delta(cl(x^l), cl(x^{-l})) = 1 + \dfrac{p^2-1}{q}$
- $\Delta(cl(x^m), cl(x^n)) = 1$ for all $m \not\equiv n \ (mod \ q)$

Results In Semi-Direct Products

Theorem

There are infinitely many nonabelian groups with $max(\Delta) = 2$

Results In S_n

Theorem

Let $n \geq 3$. Suppose that $\alpha, \beta \in S_n$, where α is a transposition and β is a k-cycle. Then:

- $\Delta(cl(\alpha), cl(\beta)) = \lfloor \frac{k}{2} \rfloor$, if $n = k$
- $\Delta(cl(\alpha), cl(\beta)) = \lfloor \frac{k}{2} \rfloor + 1$, if $n = k + 1$
- $\Delta(cl(\alpha), cl(\beta)) = \lfloor \frac{k}{2} \rfloor + 2$, if $n \geq k + 2$

Results In S_n

Theorem

Let $n \geq 3$ and $\alpha, \beta \in S_n$, where α is a transposition and β is a product of l cycles, all of length k. Then:

- $\Delta(cl(\alpha), cl(\beta)) = \lfloor \frac{k}{2} \rfloor + 1$, if $n = kl$
- $\Delta(cl(\alpha), cl(\beta)) = \lfloor \frac{k}{2} \rfloor + 2$, if $n = kl + 1$
- $\Delta(cl(\alpha), cl(\beta)) = \lfloor \frac{k}{2} \rfloor + 3$, if $n \geq kl + 2$

Results In S_n

Theorem

Let $n \geq 3$, $\alpha, \beta \in S_n$, and $k_1, \ldots, k_t \in \mathbb{Z}^+$ be arbitrary elements. Suppose that each k_i are distinct integers, β contains q_i cycles of lenght k_i for $i \in \{1, 2, \ldots, t\}$, and α is a transposition. The following values of $\Delta(cl(\alpha), cl(\beta))$ holds true:

- $\Delta(cl(\alpha), cl(\beta)) = \binom{t}{2} + \sum_{i=1}^{t} \lfloor \frac{k_i}{2} \rfloor + sgn(q_i - 1)$ for $n = \sum_{i=1}^{t} k_i$
- $\Delta(cl(\alpha), cl(\beta)) = t + \binom{t}{2} + \sum_{i=1}^{t} \lfloor \frac{k_i}{2} \rfloor + sgn(q_i - 1)$ for $n = 1 + \sum_{i=1}^{t} k_i$

Theorem (cont.)

- $\Delta(cl(\alpha), cl(\beta)) = 1 + t + \binom{t}{2} + \sum_{i=1}^{t} \lfloor \frac{k_i}{2} \rfloor + sgn(q_i - 1)$ for $n \geq 2 + \sum_{i=1}^{t} k_i$

Results In S_n

Theorem

Let $n \geq 3$ and $\alpha, \beta \in S_n$, where α is a 3-cycle and β is a k-cycle

- $\Delta(cl(\alpha), cl(\beta)) = [\frac{k^2}{12}] + 1$, if $n = k$
- $\Delta(cl(\alpha), cl(\beta)) = [\frac{k^2}{12}] + \lfloor \frac{k+1}{2} \rfloor$, if $n = k+1$
- $\Delta(cl(\alpha), cl(\beta)) = [\frac{k^2}{12}] + \lfloor \frac{k+1}{2} \rfloor + 1$, if $n = k+2$
- $\Delta(cl(\alpha), cl(\beta)) = [\frac{k^2}{12}] + \lfloor \frac{k+1}{2} \rfloor + 2$, if $n \geq k+3$

Conclusion

We have found partially the values for the function Δ for the following groups:

- Abelian groups
- D_{2n}
- $Q_{2^{n+1}}$
- DQ_{2^n}
- $C_p \rtimes C_q$, for $q \mid p-1$
- $C_n \rtimes C_q$, where each prime factor of n is congruent to 1 modulo q
- $C_p^n \rtimes C_q$,
- $C_p^n \times C_p$
- $C_{p^k} \rtimes C_p$
- S_n
  - $\Delta(cl((12)), cl(\alpha))\ \forall \alpha \in S_n$
  - $\Delta(cl(123)), cl(\beta))$, where β is a cycle or a disjoint product of cycles of the same lenght.

Future Research

- Investigate $C_n \rtimes C_m$ for all $m, n \in \mathbb{Z}^+$
- Characterize the groups with $max(\Delta) = 2$ and other values for $max(\Delta)$
- Find a lower bound for the values of Δ in $C_p \rtimes C_q$
- Find the value of Δ of any given k-cycle and l-cycle in S_n