Miscellaneous Topics in Group Theory, Combinatorics, and Number Theory (L. Reid)

- Given a group G, the conjugacy class of g is $\{xgx^{-1} \mid x\ G\}$. If A and B are subsets of G, then $AB = \{ab \mid a \in A \text{ and } b \in B\}$. It is well known that the product of any two conjugacy classes is a disjoint union of conjugacy classes. We propose investigating how many classes can be in that disjoint union. For example, it is known that in S_n the product of two non-trivial conjugacy classes cannot be a single conjugacy class. We would start by looking at S_n, but other groups could also be investigated.

- Given strings of length n consisting of 1, 2, … , m (with repeats allowed), how many have no non-decreasing (consecutive) strings of length k?

- In the (unfortunately) now defunct magazine *Quantum*, George Berzsenyi defined $G(n,k)$ to be $\max\{\mathrm{GCD}((x + 1)^n + k, x^n + k) \mid x \text{ is an integer}\}$. He computed $G(n,2)$ and asked if other values could be found. The author found explicit formulas for $G(n,3)$, $G(n,4)$, and $G(n,5)$. He believes that formulas for larger values of k can be found using similar techniques. One could also investigate other polynomial functions (partial results on the maximum of $\mathrm{GCD}(f(x + 1), f(x))$, where $f(x) = ax^2 + bx + c$, have been attained).

- How large the set of squares in a finite group G can be? It is known that for any positive integer n, there are groups where the fraction of group elements that are squares is $1/n$. It would be of great interest to find which rational numbers can occur as such a fraction. One may also ask the related question of under what conditions the set of squares forms a subgroup of G (for example, it always does if G is abelian).

- The author (with his colleagues) has recently been investigating the number of cyclic subgroups of a finite group relative to the order of the group. It would be of interest to investigate the analogous problem for abelian subgroups.