Additive Combinatorics (S. Senger)

One major arm of the field of additive combinatorics involves the distinction between additive and multiplicative structure. If A is a subset of a ring, let $A + A$ denote the set of sums of all of the pairs of elements in A, and let AA denote the set of products of all of the pairs of elements in A. The classical sums and products problem seeks to bounds on the minimum of $\{|A + A|, |AA|\}$ when A is a finite subset of the real numbers. When the set in question is an arithmetic progression, there will be few distinct sums, but many distinct products. In this case, we say that there is much additive structure. Conversely, when the original set is a geometric progression, it has many sums, but few products, and hence, much multiplicative structure. This problem has seen much effort over the years [1, 2, 4], and is still far from resolved.

The author has shown in [3] that an additive shift can disrupt the multiplicative structure in a product set. Explicitly, it was shown that if A is a finite subset of the real numbers, $AA+1$ must have many elements outside of any geometric progression of comparable length. A similar result is also shown for finite fields. However, in both settings, it is conjectured that this discrepancy between sets should be much greater. This problem would be great for an REU because it is simply stated, and the tools and techniques employed are mostly rather elementary.

Prerequisites: For the real setting, basic geometry and algebra will suffice. For the finite field setting, discrete mathematics or abstract algebra would help.

References

[1] Gy. Elekes, *On the number of sums and products*, Acta Arithmetica, 81 (1997), pp. 365–367.

[2] M. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Graduate Texts in Mathematics, Vol. 165, Springer-Verlag, (1996).

[3] S. Senger, *A note on the multiplicative structure of an additively shifted product set, AA+1*, Integers: The Electronic Journal of Combinatorial Number Theory, A34, (2013).

[4] J. Solymosi, *Bounding the multiplicative energy by the sumset*, Adv. Math., Volume 222, Issue 2, 1 October 2009, Pages 402–408.