Modeling uncertainty: a machine learning approach

Xingping Sun

The origin of the game Go can be traced back to ancient China more than 2,500 years ago. The rules of the game are simple: Players take turns to place black or white stones on a board. The player who captures the most territory on the board wins the game. But these simple rules embody a profound complexity of the game. The possible positions in Go is more than the total number of atoms in the universe, and more than a googol times larger than chess. As such, the game is played primarily through intuition and feel. The game's beauty, subtlety and intellectual depth have captured the human imagination for centuries. As recent as merely three years ago, experts in the field of computer science believed that in our life time, computers would not be capable of beating humans in playing the game. This conception was turned upside down in March 2016 when AlphaGo developed by Google DeepMind defeated the legendary Lee Sedol, the top Go player in the world over the past decade.

Traditional AI methods are deterministic in nature, which entail constructing a search tree over all possible positions. This approach is hopeless in playing Go. The way AlphaGo succeeded in defeating world champion Lee Sedol is a combination of machine learning algorithms and Monte Carlo tree search, which is intrinsically stochastic. In an empathetical manner, AlphaGo has given us a wake-up call: To effectively model a complex system with a high level of uncertainty, one needs to take a stochastic approach. In what follows, we will take up the problem known as "midpoint iteration of polygons", and describe both the deterministic and the stochastic versions of it.

The Deterministic Version: Let $\Pi^{(0)}$ be a closed polygon in the plane with vertices $v_0^{(0)}, \dots, v_{n-1}^{(0)}$. Denote by $v_0^{(1)}, \dots, v_{n-1}^{(1)}$ the midpoints of the edges $v_0^{(0)}v_1^{(0)}, v_1^{(0)}v_2^{(0)}, \dots, v_{n-1}^{(0)}v_0^{(0)}$, respectively. Connecting $v_0^{(1)}, v_1^{(1)}, \dots, v_{k-1}^{(1)}$ in the same order as above, we derive a new polygon, denoted by $\Pi^{(1)}$. Apply the same procedure to derive polygon $\Pi^{(2)}$. After k constructions, we obtain polygon $\Pi^{(k)}$. Show that $\Pi^{(k)}$ converges, as $k \to \infty$, to the centroid of the original points $v_0^{(0)}, v_1^{(0)}, \dots, v_{n-1}^{(0)}$. Jean Gaston Darboux [1] first proposed and solved the above problem in 1887. More generalized versions of the problem have been studied in [3], [4], [5], [6], [7], [8], [9], and the references therein.

The Stochastic Version: Let $\Pi^{(j)}$ be a closed polygon in the plane, and $z_0^{(j)}, z_1^{(j)}, \dots, z_{k-1}^{(j)}$ be its vertices. Parametrize each side $z_i^{(j)}z_{i+1}^{(j)}$ by

$$t \mapsto (1-t)z_i^{(j)} + tz_{i+1}^{(j)} \quad t \in [0,1], \quad i = 0, 1, \dots, k-1 \quad (z_k^{(j)} = z_0^{(j)}).$$

Let $g_i^{(j)}(t)$ be the density function of a naturally-selected probabilistic distribution on the side $z_{i-1}^{(j)}z_{i+1}^{(j)}$. (The usual suspects are: uniform distribution and Gaussian distribution.) Choose the new vertex $z_i^{(j+1)}$ according to this distribution. That is, the probability of choosing $z_{i-1}^{(j+1)}$ in the range $0 \le t_1 < t_2 \le 1$ is given by

$$\int_{t_1}^{t_2} g_i^{(j)}(t)dt.$$

This gives rise to a new (stochastic) polygon $\Pi^{(j+1)}$. Assume that all the random variables $z_i^{(j)}$ ($j = 0, 1, \ldots,\ i = 0, 1, \ldots, k-1$) are independent. Explore conditions under which the sequence of (stochastic) polygons $\Pi^{(n)}$ converges in probability to a single point. That is, there exists a fixed point x_0, such that, for any given $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left\{\max_{0\le i<k} \text{dist}(z_i^{(n)} - x_0) > \epsilon\right\} = 0.$$

Some stochastic polygon sequences have been utilized in designing computer games. We will s-tart with the uniform distribution and fully use the random number generator offered by Matlab. Theoretically, we will try to establish a Chebyshev type estimate. A similar problem concerning Bernstein polynomials based on scattered points has been studied in [10] and [11].

Prerequisite: A solid grasp of calculus and linear algebra concepts is required, and some familiarity with probability theory is desired.

References

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