Image Processing on Hexagonal and Triangular Domains (X. Sun)

The lattice A_d provides a fertile field for undergraduate students' research. In the mathematical literature, A_2 is often called the hexagonal lattice, and A_3 the face-centered cubic (fcc) lattice. They respectively give best packing for dimensions two and three; see Conway and Sloane [1], and Hales [2]. The Voronoi cells of A_2 are hexagons, which can be considered to be the union of the images of a regular triangle under a dihedral group. The Voronoi cells of A_3 are rhombic dodecahedrons. For general dimension d, the Voronoi cell V centered at zero can be expressed as the union of the images of the fundamental simplex under the Weyl group; see [1]. The author [10] has recently proved that V is a zonotope with (d+1) generators. This result has facilitated the study of discrete Fourier analysis and other approximation problems on the Voronoi cells of A_d [4], [5], [6], [7], [8]. The study of these problems depends on a certain structure of finite subgroups induced by refinement matrices for the lattice A_d, which consequently leads to a similar operation on the dual groups. If the refinement matrices are diagonal matrices with entry $1/n(d+1)$, where n is a natural number and is fixed, then these subgroups can also be characterized as modules. The author [9] uses a "slide show" technique to compute the orders of these subgroups and to find closed forms of the Dirichlet and the Fejer kernels. These formulas have been previously obtained in [4], [5], and [10]. However, the slide show technique is more geometrically revealing, and is more accessible to undergraduate students. Mastering this technique is important to the study of many image processing problems on hexagonal and triangular domains.

The author proposes to study the structure of these subgroups induced by more general refinement matrices. In particular, he wants to direct students to investigate refinement matrices whose diagonal entries are reciprocals of at least two distinct natural numbers. Mathematically, this explores the compatibility of a local refinement and the global zonotopal algebra on the Voronoi cells. In image processing, this is equivalent to say that the numerical algorithms have the local "zoom-in" feature.

There are many interesting questions to be answered. More importantly, students will have ample opportunities to see how linear algebra knowledge can be applied to image processing. To those students whose abstract algebra background is solid, it is delightful to explore the algebraic structure of the discrete subgroups induced by this kind of generalized refinement matrices. They will investigate whether or not the module structure still exists. They will be directed to develop more sophisticated "slide show" techniques in order to compute the orders of the subgroups. Those students will also study a variety of isomorphisms and homomorphisms among these discrete finite groups and their dual groups.

In image processing, "noise reduction" is paramount. There are many different kinds of noise, which can come from a variety of sources. Students need to understand the mathematical principle in noise reduction, especially the mathematical theory of Shannon sampling and interpolation [6]. To test the numerical algorithms' robustness against a variety of noise, the author will direct students to investigate interpolation at scattered sites, which entails departing from the lattice points. The unavailability of group structures in this setting will motivate students to absorb ideas from numerical analysis.

Prerequisites: A basic background in calculus and linear algebra, and some experience with abstract algebra and analysis.

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