GENERALIZED MATLIS DUALITY

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Abstract. Let \( R \) be a commutative noetherian ring and let \( E \) be the minimal injective cogenerator of the category of \( R \)-modules. A module \( M \) is said to be reflexive with respect to \( E \) if the natural evaluation map from \( M \) to \( \text{Hom}_R(\text{Hom}_R(M, E), E) \) is an isomorphism. We give a classification of modules which are reflexive with respect to \( E \). A module \( M \) is reflexive with respect to \( E \) if and only if \( M \) has a finitely generated submodule \( S \) such that \( M/S \) is artinian and \( R/\text{ann}(M) \) is a complete semi-local ring.

Matlis and Gabriel in [7] and [5] considered modules over a complete local ring \( R \). They showed that if the dual of an \( R \)-module is taken with respect to \( E_R(k) \) (the injective envelope of the residue field \( k \) of \( R \)), then finitely generated and artinian modules are reflexive.

Various authors have considered related questions. For example, dropping the condition that \( R \) be complete or weakening local to semilocal ([1], [2], [6], [8], [9]). In this paper we let \( R \) be any commutative noetherian ring and let \( E \) be the minimal injective cogenerator in the category of \( R \)-modules. We give a classification of modules which are reflexive with respect to \( E \). The result is that a module \( M \) is reflexive with respect to \( E \) if and only if \( M \) has a finitely generated submodule \( S \) such that \( M/S \) is artinian and \( R/\text{ann}(M) \) is a complete semi-local ring.

We denote by \( \Omega \) the maximal spectrum of \( R \), and we let \( E = \bigoplus_{m \in \Omega} E_R(R/m) \) be the minimal injective cogenerator in the category of \( R \)-modules. For an \( R \)-module \( M \) we let \( M^\vee = \text{Hom}_R(M, E) \) and call \( M^\vee \) the Matlis dual of \( M \). If the canonical map \( M \to M^{\vee\vee} \) is an isomorphism we say that \( M \) is (Matlis) reflexive. We note that for any \( M \), the map \( M \to M^{\vee\vee} \) is an injection. From this it is easy to conclude that \( \text{ann}(M) = \text{ann}(M^\vee) \).

If \( S \subset R \) is a multiplicative set and the canonical map \( M \to S^{-1}M \) is an isomorphism, we write \( M = S^{-1}M \). If \( M = S^{-1}M \), then also \( M^\vee = S^{-1}(M^\vee) \).

When \( S = R - P \) is the complement of a prime ideal \( P \) of \( R \) we use the usual notation \( M_P \).

If \( R \) is a local ring we let \( \hat{R} \) denote its completion. If \( M \) is finitely generated we note that \( \hat{R} \otimes_R M \cong \hat{M} \) (the completion of \( M \)). We write \( \hat{R} \otimes_R M \cong M = M \) to mean that \( M \to \hat{R} \otimes_R M \cong \hat{M} \) is an isomorphism.

We note that if \( m \in \Omega \) and \( M \) is a finitely generated \( R \)-module \( M \), then \( \text{Hom}_R(M, E(R/m)) \neq 0 \) if and only if \( \text{ann}(M) \subset m \).

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If $R$ is a complete local ring, then all finitely generated and all artinian $R$-modules are reflexive. If $R$ is a complete semilocal ring, then since $R$ is the product of a finite number of complete local rings, we still have finitely generated and artinian modules over $R$ are reflexive.

**Lemma 1.** If $I \subset R$ is an ideal and $IM = 0$ for an $R$-module $M$, then $M$ is reflexive as an $R$-module if and only if $M$ is reflexive as an $R/I$-module.

*Proof.* This follows from the fact that $\text{Hom}_R(R/I, E)$ is a minimal injective cogenerator over $R/I$ and that, since $M \otimes_R R/I \cong M$, we have

$$\text{Hom}_{R/I}(M, \text{Hom}_R(R/I, E)) \cong \text{Hom}_R(M, E).$$

**Lemma 2.** If $S \subset R$ is multiplicative and $S^{-1}M = M$ for an $R$-module $M$, then $M$ is reflexive as an $R$-module if and only if $M$ is reflexive as an $S^{-1}R$-module.

*Proof.* As in Lemma 1, $\text{Hom}_R(S^{-1}R, E)$ is a minimal injective cogenerator over $S^{-1}R$ and

$$\text{Hom}_{S^{-1}R}(M, \text{Hom}_R(S^{-1}R, E)) \cong \text{Hom}_R(M, E).$$

The following result strengthens Theorem 2(i) in [1].

**Theorem 3.** Let $R$ be a local ring and $M$ a finitely generated $R$-module. Then $M$ is reflexive if and only if $\hat{R} \otimes_R M = M$.

*Proof.* For any such $M$ we have the commutative diagram

$$
\begin{array}{ccc}
M & \longrightarrow & \text{Hom}_R(M, E_R(k)), E_R(k)) \\
\downarrow & & \downarrow \\
\hat{M} = \hat{R} \otimes_R M & \longrightarrow & \text{Hom}_{\hat{R}}(\hat{M}, E_{\hat{R}}(k)), E_{\hat{R}}(k))
\end{array}
$$

where $k$ is the residue field of $R$ and of $\hat{R}$. The bottom horizontal arrow is an isomorphism since $\hat{M}$ is a reflexive $\hat{R}$-module.

But we claim that the right vertical arrow is an isomorphism. For $E_R(k) = E_{\hat{R}}(k)$ and

$$\text{Hom}_{\hat{R}}(\hat{M}, E_{\hat{R}}(k)) = \text{Hom}_{\hat{R}}(\hat{R} \otimes_R M, E_{\hat{R}}(k)) = \text{Hom}_R(M, E_{\hat{R}}(k)) = \text{Hom}_R(M, E_R(k)).$$

Since $A = \text{Hom}_R(M, E_R(k))$ and $E_R(k)$ are artinian $R$ and $\hat{R}$-modules, we have $\text{Hom}_R(A, E_R(k)) = \text{Hom}_R(A, E_R(k))$. It follows that $M \rightarrow \hat{R} \otimes_R M$ is an isomorphism if and only if $M$ is reflexive.

**Corollary 4.** If $I$ is an ideal in a local ring $R$, then $R/I$ is reflexive as an $R$-module (or as an $R/I$-module) if and only if $R/I$ is a complete local ring.

*Proof.* This follows from the fact that $\hat{R} \otimes_R R/I = \hat{R}/I$.

**Lemma 5.** If $M$ is an $R$-module and $S \subset M$ is a submodule, then $M$ is reflexive if and only if $S$ and $M/S$ are reflexive.
Proof. The commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & S & \rightarrow & M & \rightarrow & M/S & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & S^\vee & \rightarrow & M^\vee & \rightarrow & (M/S)^\vee & \rightarrow & 0
\end{array}
\]

has exact rows and the vertical maps are all injections. So the result follows.

Lemma 6. No infinite direct sum of nonzero R-modules is reflexive.

Proof. Let \( \bigoplus M_i \) be reflexive where \( I \) is infinite and \( M_i \neq 0 \) for all \( i \in I \). We have

\[
(\bigoplus M_i)^\vee = \prod (M_i^\vee),
\]

and the canonical map

\[
\bigoplus M_i \rightarrow \left( \prod (M_i^\vee) \right)^\vee \cong \left( \bigoplus M_i \right)^{\vee \vee}
\]

sends \((x_i)_{i \in I} \) to the map \((\phi_i)_{i \in I} \mapsto \sum \phi_i(x_i)\). So the image of \((x_i)_{i \in I} \) is 0 on \( \bigoplus (M_i^\vee) \subseteq \prod (M_i^\vee) \) if and only if \( x_i = 0 \) for all \( i \), i.e., if and only if \( (x_i)_{i \in I} = 0 \).

But \( \bigoplus (M_i^\vee) \neq \prod (M_i^\vee) \) so there is a nonzero linear \( f : \prod (M_i^\vee) \rightarrow E \) which is 0 on \( \bigoplus (M_i^\vee) \). By the above no such \( f \) can be the image of an \((x_i)_{i \in I} \in \bigoplus M_i\) and so \( \bigoplus M_i \) is not reflexive.

Corollary 7. If \( M \) is reflexive there is a finitely generated submodule \( S \) such that \( M/S \) is artinian.

Proof. If \( M = 0 \) this is trivial. If \( M \neq 0 \) there is a finitely generated \( S_1 \subset M \) such that \( \text{Soc}(M/S_1) \neq 0 \). If \( \text{Soc}(M/S_1) \) is essential in \( M/S_1 \), it is well known that \( M/S_1 \) (and in fact \( E(M/S_1) \)) is artinian. If it is not essential, let \( N/S_1 \cap \text{Soc}(M/S_1) = 0 \) with \( S_1 \subset N, S_1 \neq N \). Then there is a finitely generated \( S_2 \) with \( S_1 \subset S_2 \subset N \) and \( \text{Soc}(N/S_2) \neq 0 \). But then \( \text{Soc}(M/S_1) \rightarrow \text{Soc}(M/S_2) \) is injective but not surjective. We repeat the procedure and see that it must stop, for otherwise if \( T = \bigcup S_n \), then \( \text{Soc}(M/T) \) is an infinite direct sum. This is impossible by Lemma 6.

The argument above is taken from Enochs [3], Proposition 1.3, and is included here for completeness.

Proposition 8. Let \( M \) be an R-module and suppose that for some \( m \in \Omega \), \( \text{Hom}_R(M, E(R/n)) = 0 \) for \( n \in \Omega, n \neq m \). Then if \( M \) is reflexive, \( M_m = M \) and \( \text{Hom}_R(M^\vee, E(R/n)) = 0 \) for \( n \neq m \).

Proof. Let \( M \) be reflexive and \( m \in \Omega \) be such that \( \text{Hom}_R(M, E(R/n)) = 0 \) for \( n \neq m \). If \( M \neq 0 \), we have a natural nonzero homomorphism \( M \rightarrow M^\vee = \text{Hom}_R(M^\vee, E) \) (\( n \in \Omega \)). So if \( \text{Hom}_R(M^\vee, E(R/n)) \neq 0 \), then the projection \( E \rightarrow E(R/n) \) induces \( \text{Hom}_R(M^\vee, E) \rightarrow \text{Hom}_R(M^\vee, E(R/n)) \) and so we have a nonzero homomorphism \( M \rightarrow \text{Hom}_R(M^\vee, E(R/n)) \). So then

\[
\text{Hom}_R(M, \text{Hom}_R(M^\vee, E(R/n))) \neq 0.
\]
But

$$\text{Hom}_R(M, \text{Hom}_R(M^\vee, E(R/n_i))) \cong \text{Hom}_R(M^\vee, \text{Hom}_R(M, E(R/n_i)))$$

so $\text{Hom}_R(M, E(R/n)) \neq 0$. Hence $n = m$.

Since $M^\vee = \text{Hom}_R(M, E(R/m))$, it follows easily using properties of $E(R/m)$ that $(M^\vee)_m = M^\vee$. But then $(M^\vee)_m = M^\vee$ and so $M_m = M$. It now follows by Lemma 2 that $M$ is a reflexive $R_m$-module as well as a reflexive $R$-module. \hfill \Box

In the next result we use the fact that if $A$ is an artinian $R$-module, then there are distinct maximal ideals $n_1, n_2, \ldots, n_t \in \Omega$ and a decomposition

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_t$$

such that $(A_i)_{n_i} = A_i$. The $n_1, \ldots, n_t$ and the decomposition are unique in the obvious sense. If $B \subset A$ is a submodule, then

$$B = (B \cap A_1) \oplus (B \cap A_2) \oplus \cdots \oplus (B \cap A_t)$$

gives the corresponding decomposition of $B$. We note that $\text{Hom}_R(A_i, E(R/m)) = 0$ for $m \in \Omega, m \notin \{n_1, \ldots, n_t\}$.

**Theorem 9.** Let $M$ be a finitely generated $R$-module and let $I = \text{ann}(M)$. Then $M$ is reflexive if and only if $R/I$ is a complete semilocal ring.

**Proof.** We have $M^\vee = \text{Hom}_R(M, \bigoplus E(R/m)) \cong \bigoplus \text{Hom}_R(M, E(R/m))$ (the direct sum over all $m \in \Omega$) since $M$ is finitely generated. Since $M^\vee$ is also reflexive, we see by Proposition 8 that $\text{Hom}_R(M, E(R/m)) = 0$ except for a finite number of $m \in \Omega$. Let $n_1, n_2, \ldots, n_t \in \Omega$ be distinct elements of $\Omega$ such that $\text{Hom}_R(M, E(R/m)) = 0$ for $m \notin \{n_1, n_2, \ldots, n_t\}$, so $M^\vee = \bigoplus_{i=1}^t \text{Hom}_R(M, E(R/n_i))$. We can assume $\text{Hom}_R(M, E(R/n_i)) \neq 0$ for $i = 1, 2, \ldots, t$. If $M_i$ denotes $\text{Hom}_R(M, E(R/n_i))$, then since $\text{Hom}_R(M, E(R/m)) = 0$ for $m \neq n_i$, we see $\text{Hom}_R(R/I_i, E(R/m)) = 0$ for $m \neq n_i$ where $I_i = \text{ann}(M_i)$. Hence $I_i$ is contained in only one maximal ideal, namely $n_i$. Hence $R/I_i$ is a local ring and $(R/I_i)_{n_i} = R/I_i$.

Since $R/I_i$ is also reflexive and finitely generated, we get that $R/I_i$ is a reflexive $R_{n_i}$-module by Lemma 2. But then by Theorem 9, $R_{n_i} \otimes_{R_{n_i}} R/I_i = R/I_i$. This means that $R/I_i$ is a complete local ring.

But now if $I = \text{ann}(M)$, we have $I = \bigcap_{i=1}^t I_i$. The $I_1, \ldots, I_t$ are pairwise comaximal so by the Chinese remainder theorem $R/I \cong R/I_1 \times R/I_2 \times \cdots \times R/I_t$. Hence $R/I$ is a complete semilocal ring.

Now assume that $M$ is a finitely generated $R$-module and that $R/I$ is a complete semilocal ring where $I = \text{ann}(M)$. Then $M$ is reflexive as an $R/I$-module and so by Lemma 1 is reflexive as an $R$-module. \hfill \Box

**Corollary 10.** If $I \subset R$ is an ideal, then $R/I$ is reflexive if and only if $R/I$ is a complete semilocal ring.

**Proof.** Immediate. \hfill \Box

**Corollary 11.** If $I, J \subset R$ are ideals such that $R/I$ and $R/J$ are complete semilocal rings, then $R/IJ$ is a complete semilocal ring.
Proof. We only need argue that \( R/IJ \) is a reflexive \( R \)-module. We have the exact sequence
\[
0 \rightarrow I/IJ \rightarrow R/IJ \rightarrow R/I \rightarrow 0.
\]
But \( R/I \) is a reflexive \( R \)-module, and since \( I/IJ \) is a quotient of \((R/J)^n\) for some \( n \geq 1 \) it is a reflexive \( R \)-module. Hence \( R/IJ \) is a reflexive \( R \)-module. (These claims all use Lemma 5.)

We now give the complete classification of reflexive modules.

**Theorem 12.** An \( R \)-module \( M \) is reflexive if and only if it has a finitely generated submodule \( S \) such that \( M=S \) is artinian and if \( R/I \) is a complete semilocal ring where \( I=\text{ann}(M) \).

**Proof.** Assume that \( M \) is reflexive. By Corollary 7 we have the finitely generated \( S \) with \( M=S \) artinian. Since \( S \) is reflexive and finitely generated, \( R/J \) is a complete semilocal ring where \( J=\text{ann}(S) \) by Theorem 9. Since \( M=S \) is artinian and reflexive, \( R/K \) is complete semilocal. But \( \text{ann}(M/S)^\vee=\text{ann}(M/S) \). Since \( I=\text{ann}(M) \supset JK \) and since \( R/JK \) is complete semilocal by Corollary 11 we see that \( R/I \) is complete semilocal.

Conversely assume that \( M \) has a finitely generated submodule \( S \subset M \) with \( M=S \) artinian and that \( R/I \) is a complete semilocal ring where \( I=\text{ann}(M) \).

Then, since \( J=\text{ann}(S) \supset \text{ann}(M) = I \), we see that \( R/J \) is complete and semilocal. Hence \( S \) is reflexive.

Since \( M/S \) is artinian and \( K=\text{ann}(M/S) \supset \text{ann}(M) = I \), we see that \( R/K \) is complete and semilocal. Then the artinian \( R/K \)-module is reflexive as an \( R/K \)-module. By Lemma 11 this implies that \( M/S \) is a reflexive \( R \)-module. But then by an appeal to Lemma 5 it follows that \( M \) is a reflexive \( R \)-module.

**Examples.**

1. If \( k \) is an algebraically closed field and \( I \subset k[x_1, \ldots, x_n] \) is such that \( k[x_1, \ldots, x_n]/I \) is semilocal, then in fact \( k[x_1, \ldots, x_n]/I \) is artinian. Hence a finitely generated reflexive module \( M \) over \( k[x_1, \ldots, x_n] \) has finite length. Since the dual \( A^\vee \) of an artinian reflexive \( k[x_1, \ldots, x_n] \)-module \( A \) is finitely generated, we see that \( A \) also has finite length. So by Theorem 12 the reflexive \( k[x_1, \ldots, x_n] \)-modules are exactly those of finite length.

2. If \( k \) is any field and \( R=k[[x]][y] \), then \( R/(y) \) is a complete local ring and so is a reflexive \( R \)-module which is not of finite length.

**Remark.** We recall that a module \( M \) is said to be cotorsion if \( xt^1(F, M) = 0 \) for all flat \( R \)-modules \( F \). In [4], finitely generated cotorsion modules were characterized over a commutative noetherian ring \( R \) of finite Krull dimension. In effect, over such a ring \( R \), a finitely generated module \( M \) is cotorsion if and only if it is reflexive. We do not know if this holds if we allow \( R \) to have infinite Krull dimension.

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