Some change of ring theorems for matlis reflexive modules

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SOME CHANGE OF RING THEOREMS FOR MATLIS REFLEXIVE MODULES

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Abstract. Suppose \((R, m) \rightarrow (S, n)\) is a local homomorphism of local rings. We show that if \(M\) is a Matlis reflexive \(R\)-module, then \(\text{Ext}^i_R(S, M)\) and \(\text{Tor}^i_S(S, M)\) are Matlis reflexive \(S\)-modules if \(S\) is module-finite over the image of \(R\). In case \(S = \hat{R}\), the \(m\)-adic completion of \(R\), we show that if \(M\) is a reflexive \(R\)-module, then \(\hat{R} \otimes_R M\) is a reflexive \(\hat{R}\)-module and in fact \(M \cong \hat{R} \otimes_R M\). We also show that if \(R\) is any local ring and \(M\) and \(N\) are two reflexive \(R\)-modules, then \(\text{Ext}^i_R(M, N)\) and \(\text{Tor}^i_R(M, N)\) are reflexive \(R\)-modules for all \(i\).

A ring will always mean a commutative, noetherian ring with identity. If \(R\) is a ring and \(M\) an \(R\)-module then \(E_R(M)\) will denote an injective envelope (or injective hull) of \(M\). If \((R, m)\) is local and \(E = E_R(R/m)\), then \(D(\_\_\_)\) will denote the exact contravariant functor \(\text{Hom}_R(\_, E)\), called the Matlis dual functor, and an \(R\)-module \(M\) is said to be Matlis reflexive if the canonical injection \(M \rightarrow D(D(M))\) is an isomorphism. Throughout this paper “reflexive” will mean Matlis reflexive, and “dual” will mean Matlis dual. Results from Matlis [5] and Bass [1] will be freely used. For general references, see also [6, §18], [8, Chapter 3], and [9].

One aim of this note is to extend and clarify some of the results in [2]. For example, it was shown in [2, Theorem 4, p.1110] that if \(R\) is a complete local ring and \(M\) and \(N\) are reflexive \(R\)-modules, then \(\text{Hom}_R(M, N), M \otimes_R N,\)
and also \( \text{Ext}_i^R(M, N) \) and \( \text{Tor}_i^R(M, N) \) for \( i \geq 1 \) are reflexive \( R \)-modules. Furthermore, we have the following "ext-tor" duality for reflexive modules: 
\[
D(\text{Ext}_i^R(M, N)) \cong \text{Tor}_i^R(M, D(N)).
\]
In this note, we show that it is not necessary to assume that \( R \) is complete.

Before we state our next result we introduce some notation that will be used throughout this paper. If \((R, m) \to (S, n)\) is a local homomorphism of local rings, then \( D(\_\_) \) will denote the Matlis dual functor \( \text{Hom}_R(\_\_, E) \), where \( E = E_R(R/m) \), and \( D'(\_\_) \) will denote \( \text{Hom}_S(\_\_, E') \), where \( E' = E_S(S/n) \).

We first consider the case where the local homomorphism \( R \to S \) makes \( S \) a finitely generated \( R \)-module, an obvious example being the canonical map \( R \to R/I \), where \( I \) is an ideal of \( R \).

**Theorem 1** Let \((R, m) \to (S, n)\) be a local homomorphism of local rings, and suppose that \( S \) is module-finite over the image of \( R \). Let \( M \) be an \( R \)-module. Then, using the notation above,

1. \( D'(\text{Ext}_i^R(S, M)) \cong \text{Tor}_i^R(S, D(M)) \),
2. \( D'(\text{Tor}_i^R(S, M)) \cong \text{Ext}_i^R(S, D(M)) \),
3. If \( M \) is a reflexive \( R \)-module, then \( \text{Ext}_i^R(S, M) \) and \( \text{Tor}_i^R(S, M) \) are reflexive \( S \)-modules for all \( i \geq 0 \).

In Section 2, we will consider the local homomorphism \( R \to \hat{R} \), where \( \hat{R} \) is the \( m \)-adic completion of \( R \).

**Theorem 2** Let \( R \) be a local ring and let \( M \) be a reflexive \( R \)-module. Then

1. \( M \cong \hat{R} \otimes_R M \) as \( R \)-modules.
2. \( M \) has a finitely generated submodule \( S \) which is complete in the \( m \)-adic topology, such that \( M/S \) is artinian.
3. \( \hat{R} \otimes_R M \) is a reflexive \( \hat{R} \)-module.

In Section 3 we prove the following result, which, as mentioned above, is already known in the case where \( R \) is a complete local ring.

**Theorem 3** Let \( R \) be any local ring. Let \( M \) and \( N \) be reflexive \( R \)-modules. Then \( \text{Ext}_i^R(M, N) \) and \( \text{Tor}_i^R(M, N) \) are reflexive \( R \)-modules for all \( i \geq 0 \) and
\[
D(\text{Ext}_i^R(M, N)) \cong \text{Tor}_i^R(M, D(N)),
\]
\[
D(\text{Tor}_i^R(M, N)) \cong \text{Ext}_i^R(M, D(N)).
\]

Although Theorem 3 is not a "change of ring" theorem, it can be proved using Theorem 2 and the result in [2] cited above. However, we will give a direct proof of Theorem 3.
1 The proof of Theorem 1

Lemma 1 Let \((R, m)\) be a local ring. If \(M\) is any finitely generated \(R\)-module, and \(N\) is reflexive, then \(\text{Ext}_R^i(M,N)\) and \(\text{Tor}_R^i(M,N)\) are reflexive \(R\)-modules for all \(i \geq 0\) and

\[
D(\text{Ext}_R^i(M,N)) \cong \text{Tor}_R^i(M,D(N)), \\
D(\text{Tor}_R^i(M,N)) \cong \text{Ext}_R^i(M,D(N)).
\]

Proof. The isomorphisms are the usual “ext-tor dualities” (see Strooker [8], Proposition 3.4.14 (ii), p.47). In fact, the second isomorphism holds without any hypotheses on \(M\) or \(N\) (by [8, Proposition 3.4.14(i)]). By taking duals and using the fact that \(N\) is reflexive it follows that \(\text{Ext}_R^i(M,N)\) and \(\text{Tor}_R^i(M,N)\) are reflexive. \(\Box\)

Proof of Theorem 1. It is easy to see that \(\text{Hom}_R(S,E)\) is an injective \(S\)-module, and it is known (by Hochster’s notes [4]) that \(\text{Hom}_R(S,E)\) is an injective envelope of \(S/n\) in case \(S\) is a finitely generated \(R\)-module. Thus we may let \(E' = \text{Hom}_R(S,E)\). It then follows immediately that for any \(S\)-module \(N\), \(\text{Hom}_S(N,E') \cong \text{Hom}_R(N \otimes_S S, E)\), and so there is a natural isomorphism of functors (of \(S\)-modules)

\[
D'(-) \cong D(- \otimes_S S).
\]

To prove (1), we have \(D'(\text{Ext}_R^i(S,M)) \cong D(\text{Ext}_R^i(S,M) \otimes_S S)\). But since \(S\) is finitely generated, it follows by the “ext-tor” duality cited above that there is an isomorphism \(D(\text{Ext}_R^i(S,M)) \cong \text{Tor}_R^i(S,D(M))\). To prove (2), we have

\[
D'(\text{Tor}_R^i(S,M)) \cong D(\text{Tor}_R^i(S,M)) \cong \text{Ext}_R^i(S,D(M)).
\]

Finally (3) follows immediately from (1) and (2) by taking duals. \(\Box\)

Next we discuss an example which shows that Theorem 1 is not necessarily true if \(S\) is not finitely generated as an \(R\)-module. We will need to use the fact that if \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) is exact, then \(B\) is reflexive if and only if \(A\) and \(C\) are reflexive. This follows by mapping the short exact sequence into its double dual and applying the snake lemma.

First of all, it is clear that any field \(k\) is a reflexive \(k\)-module. In fact, in this case \(E_k(k) = k\), so the Matlis dual is the usual vector space dual. When \(R = k\) and \(S = k[x]/(x)\), there is an obvious local homomorphism \(R \rightarrow S\), and \(S\) is not finitely generated as an \(R\)-module.
Example 1 Suppose $k$ is a field, and $S = k[x]$. Then $\text{Hom}_k(S, k)$ is not a reflexive $S$-module.

For the inclusion map $k[x] \hookrightarrow k[x]_e$ induces a $k[x]$-linear surjection

$$\text{Hom}_k(k[x]_e, k) \twoheadrightarrow \text{Hom}_k(k[x], k),$$

and by Northcott [7, p.291] the latter is isomorphic to $k[[x^{-1}]]$ as a $k[x]$-module. However if $f(x) \notin (x)$ then $k[[x^{-1}]] \xrightarrow{f(x)} k[[x^{-1}]]$ is an isomorphism, and so $k[[x^{-1}]]$ is also an $S$-module. Hence there is an $S$-linear surjection of $S$-modules

$$\text{Hom}_k(S, k) \twoheadrightarrow k[[x^{-1}]],$$

and so it is enough to show that $k[[x^{-1}]]$ is not $S$-reflexive. But $k[x^{-1}] \subset k[[x^{-1}]]$ is an $S$-submodule, so it suffices to show that $k[x^{-1}]$ (or it’s dual) is not $S$-reflexive. By [7, Theorem 2, p.292], we know that $E_S(k) \cong k[x^{-1}]$, and so it follows from [5, Theorem 3.7, p.522] that

$$\text{Hom}_S(k^{-1}, k[x^{-1}]) \cong k[[x]].$$

Therefore the dual of $k[x^{-1}]$ is isomorphic to $\tilde{S} = k[[x]]$. But if $S$ is a local ring which is not complete, then $\tilde{S}$ is not $S$-reflexive.

2 The proof of Theorem 2

In this section we will let $R$ be local, and consider the local homomorphism $R \rightarrow \hat{R}$. Recall that if $R$ is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, then the $\mathfrak{m}$-adic completion $\hat{R}$ has the same residue field. It is also well-known that $E_R(k) \cong E_{\hat{R}}(k)$ (see [8, Theorem 3.4.8], [5], or [6, Theorem 18.6 (iii)] for details), which means, using the notation introduced earlier, that $E \cong E'$.

Proof of Theorem 2. Any artinian module $A$ embeds in a finite direct sum of copies of $E$. By taking the dual of such an embedding we see that $D(A)$ is a finitely generated $\hat{R}$-module, and hence $D(A)$ is complete.

And if $T$ is any finitely generated submodule of $M$, then $D(T)$ is artinian, so $D(D(T))$ is complete. But any submodule of a reflexive module is reflexive, so $T \cong D(D(T))$. Hence $T$ is complete. But $T \cong \hat{R} \otimes_R T$ canonically for a finitely generated $T$. So $T \rightarrow \hat{R} \otimes_R M$ is an isomorphism. Then taking the obvious direct limit of the $T$’s gives $M \rightarrow \hat{R} \otimes_R M$ is an isomorphism. This proves (i).
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If $M$ is a reflexive $R$-module, then $M$ has a finitely generated submodule $S$ such that $M/S$ is artinian. (See the proof of Proposition 1.3 in Enochs [3], or [8, Theorem 3.4.13, p.45].) There it is assumed that the ring is complete, and shown that a module $M$ is reflexive if and only if $M$ has a finitely generated submodule $S$ such that $M/S$ is artinian, but the proof of the implication we need does not require that $R$ be complete.) By the remark above, any finitely generated submodule of a reflexive module is complete, and so we have (ii).

To prove (iii), since $D'(\hat{R} \otimes_R M) \cong D(M)$ we get that

$$D(D(M)) = \text{Hom}_R(D(M), E) \quad \text{and} \quad D'(D'(\hat{R} \otimes_R M)) \cong \text{Hom}_R(D(M), E).$$

Any $\hat{R}$-linear map is also $R$-linear, and therefore the following sequence is exact: $0 \to D'(\hat{R} \otimes_R M) \to D(D(M))$. So we have a commutative diagram:

$$
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \\
0 & \to & \hat{R} \otimes_R M & \to & D'(\hat{R} \otimes_R M)
\end{array}
$$

Now, if the top row $M \to D(D(M))$ is an isomorphism, then by diagram chasing, so is the bottom row. \(\square\)

Remark. If $A$ is any artinian $R$-module, then every element of $A$ is annihilated by a power of $m$. It follows that the canonical injection $A \to \hat{R} \otimes_R A$ is also surjective. This can be used to give another proof of (i). For suppose $M$ is reflexive. Then let $S$ be a finitely generated complete submodule with $A := M/S$ artinian. Since $S \to \hat{R} \otimes_R S$ and $A \to \hat{R} \otimes_R A$ are isomorphisms, $M \to \hat{R} \otimes_R M$ is also an isomorphism.

3 The proof of Theorem 3

Throughout this section we will assume that $(R, m)$ is a local ring with residue field $k = R/m$, and $E = E_R(k)$. We will need to use the fact (see [5]) that every injective module is a direct sum of indecomposable injective modules, and each indecomposable injective module is isomorphic to $E(R/p)$ for some prime ideal $p$ of $R$.

Lemma 2 If $N$ is a reflexive $R$-module, then the Bass numbers $\mu_i(N)$ are finite for all $i$. 
Proof. Recall that if $0 \rightarrow N \rightarrow E^0(N) \rightarrow E^1(N) \rightarrow \cdots$ is a minimal injective resolution of $N$, then $\mu_i(N)$ is the number of copies of $E$ appearing in a direct sum decomposition of $E^i(N)$ into indecomposable injective modules. Let $S$ be a finitely generated submodule of $N$ with $N/S$ artinian. Then it is well-known that $\mu_i(S) < \infty$ for all $i$ (by [1, Lemma 2.7]), and it is easy to see that $\mu_i(A) < \infty$ for all $i$ for any artinian module $A$. For in this case each $E^i(A)$ is a finite direct sum of copies of $E$. But we also know that $\mu_i(N)$ is the dimension of the $k$-vector space $\text{Ext}^i(k, N)$, i.e. $\mu_i(N) = \dim_k \text{Ext}^i(k, N)$.

(See [6, Theorem 18.7], for example.) The result then follows from the long exact sequence

$$\cdots \rightarrow \text{Ext}^i(k, S) \rightarrow \text{Ext}^i(k, N) \rightarrow \text{Ext}^i(k, N/S) \rightarrow \cdots$$

and the fact that both $\text{Ext}^i(k, S)$ and $\text{Ext}^i(k, N/S)$ are finite dimensional. \(\square\)

**Lemma 3** Assume that $A$ is an artinian reflexive module and that $N$ is reflexive. Then $\text{Ext}^i_R(A, N)$ and $\text{Tor}^i_R(A, N)$ are reflexive for all $i$ and $D(\text{Ext}^i_R(A, N)) \cong \text{Tor}^i_R(A, D(N))$.

**Proof.** Let $0 \rightarrow N \rightarrow E^0(N) \rightarrow E^1(N) \rightarrow \cdots$ be a minimal injective resolution of $N$, and let $E^\bullet$ be the complex $0 \rightarrow E^0(N) \rightarrow E^1(N) \rightarrow \cdots$. Let $D^\bullet$ be the subcomplex of $E^\bullet$ with $D^i \subset E^i(N)$ being all $x$ annihilated by some power of $m$. It is easy to verify that $D^\bullet$ is a subcomplex; in fact, $D^i$ is just the sum of the $E$'s in a decomposition of $E^i(N)$ into $E(R/p)^\cdot$s for prime ideals $p$. We have an exact sequence of complexes of injective modules $0 \rightarrow D^\bullet \rightarrow E^\bullet \rightarrow E^\bullet/D^\bullet \rightarrow 0$ which splits at each level. So the sequence

$$0 \rightarrow \text{Hom}(A, D^\bullet) \rightarrow \text{Hom}(A, E^\bullet) \rightarrow \text{Hom}(A, E^\bullet/D^\bullet) \rightarrow 0$$

is exact. Since $A$ is artinian, $\text{Hom}(A, E(R/p)) = 0$ for $p \neq m$. This follows because some power of $m$ kills each element of $A$, yet if $x \notin p$ then $x$ acts as an automorphism on $E(R/p)$. Therefore, the complex $\text{Hom}(A, E^\bullet/D^\bullet)$ is zero and

$$H^i(\text{Hom}(A, D^\bullet)) \cong H^i(\text{Hom}(A, E^\bullet)) \cong \text{Ext}^i(A, N).$$

Since $N$ is reflexive, by Lemma 2 each $D^i$ is a direct sum of a finite number of copies of $E$. Hence each $\text{Hom}(A, D^i)$ is reflexive, since each $\text{Hom}(A, D^i)$ is a direct sum of a finite number of copies of $D(A)$ and $A$ is assumed to be reflexive. Thus $\text{Hom}(A, D^\bullet)$ is a complex of reflexive modules, so it’s homology is reflexive for each $i$. Therefore, $\text{Ext}^i(A, N)$ is reflexive.
Now a module is reflexive if and only if its dual is reflexive. So $D(N)$ is reflexive and the isomorphism $D(Tor_i(A,N)) \cong \text{Ext}^i(A,D(N))$ implies that $Tor_i(A,N)$ is reflexive. Finally, use this isomorphism once more: we know $D(Tor_i(A,D(N))) \cong \text{Ext}^i(A,DD(N)) \cong \text{Ext}^i(A,N)$. So take duals and use the fact that $Tor_i(A,D(N))$ is reflexive to get $D(\text{Ext}^i(A,N)) \cong Tor_i(A,D(N))$. □

Proof of Theorem 3. If $M$ is reflexive, it has a finitely generated (complete) submodule $S$ such that $M/S$ is artinian. Since $M$ is reflexive, both $S$ and $M/S$ are reflexive. Consider the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
D(\text{Ext}(S,N)) & \rightarrow & D(\text{Ext}(M,N)) & \rightarrow & D(\text{Ext}(M/S,N)) & \rightarrow \\
\uparrow & & \uparrow & & \uparrow & \\
\rightarrow & & \rightarrow & & \rightarrow & \\
\end{array}
$$

The theorem follows since the two outside maps are isomorphisms of reflexive modules. □

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References


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