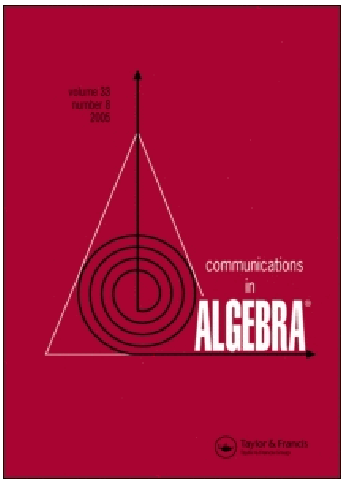

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Some change of ring theorems for matlis reflexive modules

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SOME CHANGE OF RING THEOREMS FOR MATLIS REFLEXIVE MODULES

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Abstract. Suppose $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism of local rings. We show that if M is a Matlis reflexive R-module, then $\operatorname{Ext}^i_R(S, M)$ and $\operatorname{Tor}^R_i(S, M)$ are Matlis reflexive S-modules if S is module-finite over the image of R. In case $S = \hat{R}$, the $\mathfrak{m}$-adic completion of R, we show that if M is a reflexive R-module, then $\hat{R} \otimes_R M$ is a reflexive $\hat{R}$-module and in fact $M \cong \hat{R} \otimes_R M$. We also show that if R is any local ring and M and N are two reflexive R-modules, then $\operatorname{Ext}^i_R(M, N)$ and $\operatorname{Tor}^R_i(M, N)$ are reflexive R-modules for all i.

A ring will always mean a commutative, noetherian ring with identity. If R is a ring and M an R-module then $E_R(M)$ will denote an injective envelope (or injective hull) of M. If $(R, \mathfrak{m})$ is local and $E = E_R(R/\mathfrak{m})$, then $\mathrm{D}(\_\_)$ will denote the exact contravariant functor $\operatorname{Hom}_R(\_\_, E)$, called the Matlis dual functor, and an R-module M is said to be *Matlis reflexive* if the canonical injection $M \longrightarrow \mathrm{D}(\mathrm{D}(M))$ is an isomorphism. Throughout this paper "reflexive" will mean Matlis reflexive, and "dual" will mean Matlis dual. Results from Matlis [5] and Bass [1] will be freely used. For general references, see also [6, §18], [8, Chapter 3], and [9].

One aim of this note is to extend and clarify some of the results in [2]. For example, it was shown in [2, Theorem 4, p.1110] that if R is a *complete* local ring and M and N are reflexive R-modules, then $\operatorname{Hom}_R(M, N)$, $M \otimes_R N$, and also $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tor}^R_i(M,N)$ for $i \geq 1$ are reflexive R-modules. Furthermore, we have the following "ext-tor" duality for reflexive modules: $\mathrm{D}(\mathrm{Ext}^i_R(M,N)) \cong \mathrm{Tor}^R_i(M,\mathrm{D}(N))$. In this note, we show that it is not necessary to assume that R is complete.

Before we state our next result we introduce some notation that will be used throughout this paper. If $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a local homomorphism of local rings, then $\mathrm{D}(\_\_)$ will denote the Matlis dual functor $\mathrm{Hom}_R(\_\_, E)$, where $E = E_R(R/\mathfrak{m})$, and $\mathrm{D}'(\_\_)$ will denote $\mathrm{Hom}_S(\_\_, E')$, where $E' = E_S(S/\mathfrak{n})$.

We first consider the case where the local homomorphism $R \to S$ makes S a finitely generated R-module, an obvious example being the canonical map $R \to R/I$, where I is an ideal of R.

Theorem 1 *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local homomorphism of local rings, and suppose that S is module-finite over the image of R. Let M be an R-module. Then, using the notation above,*

(1) $D'(Ext^i_R(S,M)) \cong Tor^R_i(S,D(M))$,

(2) $D'(Tor^R_i(S,M)) \cong Ext^i_R(S,D(M))$,

(3) *If M is a reflexive R-module, then $Ext^i_R(S,M)$ and $Tor^R_i(S,M)$ are reflexive S-modules for all $i \geq 0$.*

In Section 2, we will consider the local homomorphism $R \to \hat{R}$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R.

Theorem 2 *Let R be a local ring and let M be a reflexive R-module. Then*

(i) *$M \cong \hat{R} \otimes_R M$ as R-modules.*

(ii) *M has a finitely generated submodule S which is complete in the $\mathfrak{m}$-adic topology, such that M/S is artinian.*

(iii) *$\hat{R} \otimes_R M$ is a reflexive $\hat{R}$-module.*

In Section 3 we prove the following result, which, as mentioned above, is already known in the case where R is a *complete* local ring.

Theorem 3 *Let R be any local ring. Let M and N be reflexive R-modules. Then $Ext^i_R(M,N)$ and $Tor^R_i(M,N)$ are reflexive R-modules for all $i \geq 0$ and*

$$D(Ext^i_R(M,N)) \cong Tor^R_i(M,D(N)),$$

$$D(Tor^R_i(M,N)) \cong Ext^i_R(M,D(N)).$$

Although Theorem 3 is not a "change of ring" theorem, it can be proved using Theorem 2 and the result in [2] cited above. However, we will give a direct proof of Theorem 3.

1 The proof of Theorem 1

Lemma 1 *Let $(R, \mathfrak{m})$ be a local ring. If M is any finitely generated R-module, and N is reflexive, then $Ext^i_R(M,N)$ and $Tor^R_i(M,N)$ are reflexive R-modules for all $i \geq 0$ and*

$$D(Ext^i_R(M,N)) \cong Tor^R_i(M, D(N)),$$

$$D(Tor^R_i(M,N)) \cong Ext^i_R(M, D(N)).$$

Proof. The isomorphisms are the usual "ext-tor dualities" (see Strooker [8], Proposition 3.4.14 (ii), p.47). In fact, the second isomorphism holds without any hypotheses on M or N (by [8, Proposition 3.4.14(i)]). By taking duals and using the fact that N is reflexive it follows that $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tor}^R_i(M,N)$ are reflexive. □

Proof of Theorem 1. It is easy to see that $\mathrm{Hom}_R(S,E)$ is an injective S-module, and it is known (by Hochster's notes [4]) that $\mathrm{Hom}_R(S,E)$ is an injective envelope of $S/\mathfrak{n}$ in case S is a finitely generated R-module. Thus we may let $E' = \mathrm{Hom}_R(S,E)$. It then follows immediately that for any S-module N, $\mathrm{Hom}_S(N,E') \cong \mathrm{Hom}_R(N \otimes_S S, E)$, and so there is a natural isomorphism of functors (of S-modules)

$$\mathrm{D}'(\_\_) \cong \mathrm{D}(\_\_ \otimes_S S).$$

To prove (1), we have $\mathrm{D}'(\mathrm{Ext}^i_R(S,M)) \cong \mathrm{D}(\mathrm{Ext}^i_R(S,M) \otimes_S S)$. But since S is finitely generated, it follows by the "ext-tor" duality cited above that there is an isomorphism $\mathrm{D}(\mathrm{Ext}^i_R(S,M)) \cong \mathrm{Tor}^R_i(S, \mathrm{D}(M))$. To prove (2), we have

$$\mathrm{D}'(\mathrm{Tor}^R_i(S,M)) \cong \mathrm{D}(\mathrm{Tor}^R_i(S,M)) \cong \mathrm{Ext}^i_R(S, \mathrm{D}(M)).$$

Finally (3) follows immediately from (1) and (2) by taking duals. □

Next we discuss an example which shows that Theorem 1 is not necessarily true if S is not finitely generated as an R-module. We will need to use the fact that if $0 \to A \to B \to C \to 0$ is exact, then B is reflexive if and only if A and C are reflexive. This follows by mapping the short exact sequence into its double dual and applying the snake lemma.

First of all, it is clear that any field k is a reflexive k-module. In fact, in this case $E_k(k) = k$, so the Matlis dual is the usual vector space dual. When $R = k$ and $S = k[x]_{(x)}$, there is an obvious local homomorphism $R \to S$, and S is not finitely generated as an R-module.

Example 1 *Suppose k is a field, and $S = k[x]_{(x)}$. Then $Hom_k(S,k)$ is not a reflexive S-module.*

For the inclusion map $k[x] \hookrightarrow k[x]_{(x)}$ induces a $k[x]$-linear surjection

$$\mathrm{Hom}_k(k[x]_{(x)}, k) \longrightarrow \mathrm{Hom}_k(k[x], k),$$

and by Northcott [7, p.291] the latter is isomorphic to $k[[x^{-1}]]$ as a $k[x]$-module. However if $f(x) \notin (x)$ then $k[[x^{-1}]] \xrightarrow{f(x)} k[[x^{-1}]]$ is an isomorphism, and so $k[[x^{-1}]]$ is also an S-module. Hence there is an S-linear surjection of S-modules

$$\mathrm{Hom}_k(S, k) \longrightarrow k[[x^{-1}]],$$

and so it is enough to show that $k[[x^{-1}]]$ is not S-reflexive. But $k[x^{-1}] \subset k[[x^{-1}]]$ is an S-submodule, so it suffices to show that $k[x^{-1}]$ (or it's dual) is not S-reflexive. By [7, Theorem 2, p.292], we know that $E_S(k) \cong k[x^{-1}]$, and so it follows from [5, Theorem 3.7, p.522] that

$$\mathrm{Hom}_S(k[x^{-1}], k[x^{-1}]) \cong k[[x]].$$

Therefore the dual of $k[x^{-1}]$ is isomorphic to $\hat{S} = k[[x]]$. But if S is a local ring which is not complete, then $\hat{S}$ is not S-reflexive.

2 The proof of Theorem 2

In this section we will let R be local, and consider the local homomorphism $R \to \hat{R}$. Recall that if R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, then the $\mathfrak{m}$-adic completion $\hat{R}$ has the same residue field. It is also well-known that $E_R(k) \cong E_{\hat{R}}(k)$ (see [8, Theorem 3.4.8], [5], or [6, Theorem 18.6 (iii)] for details), which means, using the notation introduced earlier, that $E \cong E'$.

Proof of Theorem 2. Any artinian module A embeds in a finite direct sum of copies of E. By taking the dual of such an embedding we see that $\mathrm{D}(A)$ is a finitely generated $\hat{R}$-module, and hence $\mathrm{D}(A)$ is complete.

And if T is any finitely generated submodule of M, then $\mathrm{D}(T)$ is artinian, so $\mathrm{D}(\mathrm{D}(T))$ is complete. But any submodule of a reflexive module is reflexive, so $T \cong \mathrm{D}(\mathrm{D}(T))$. Hence T is complete. But $\hat{T} \cong \hat{R} \otimes_R T$ canonically for a finitely generated T. So $T \to \hat{R} \otimes_R T$ is an isomorphism. Then taking the obvious direct limit of the T's gives $M \to \hat{R} \otimes_R M$ is an isomorphism. This proves (i).

If M is a reflexive R-module, then M has a finitely generated submodule S such that M/S is artinian. (See the proof of Proposition 1.3 in Enochs [3], or [8, Theorem 3.4.13, p.45].) There it is assumed that the ring is complete, and shown that a module M is reflexive if and only if M has a finitely generated submodule S such that M/S is artinian, but the proof of the implication we need does not require that R be complete.) By the remark above, any finitely generated submodule of a reflexive module is complete, and so we have (ii).

To prove (iii), since $\mathrm{D}'(\hat{R}\otimes_R M) \cong \mathrm{D}(M)$ we get that

$$\mathrm{D}(\mathrm{D}(M)) = \mathrm{Hom}_R(\mathrm{D}(M), E) \quad \text{and} \quad \mathrm{D}'(\mathrm{D}'(\hat{R}\otimes_R M)) \cong \mathrm{Hom}_{\hat{R}}(\mathrm{D}(M), E).$$

Any $\hat{R}$-linear map is also R-linear, and therefore the following sequence is exact: $0 \to \mathrm{D}'(\mathrm{D}'(\hat{R}\otimes_R M)) \to \mathrm{D}(\mathrm{D}(M))$. So we have a commutative diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathrm{D}(\mathrm{D}(M)) \\ & & \downarrow & & \uparrow \\ 0 & \longrightarrow & \hat{R}\otimes_R M & \longrightarrow & \mathrm{D}'\mathrm{D}'(\hat{R}\otimes_R M)) \\ & & & & \uparrow \\ & & & & 0 \end{array}$$

Now, if the top row $M \to \mathrm{D}(\mathrm{D}(M))$ is an isomorphism, then by diagram chasing, so is the bottom row. □

Remark. If A is any artinian R-module, then every element of A is annihilated by a power of $\mathfrak{m}$. It follows that the canonical injection $A \to \hat{R}\otimes_R A$ is also surjective. This can be used to give another proof of (i). For suppose M is reflexive. Then let S be a finitely generated complete submodule with $A := M/S$ artinian. Since $S \to \hat{R}\otimes_R S$ and $A \to \hat{R}\otimes_R A$ are isomorphisms, $M \to \hat{R}\otimes_R M$ is also an isomorphism.

3 The proof of Theorem 3

Throughout this section we will assume that $(R, \mathfrak{m})$ is a local ring with residue field $k = R/\mathfrak{m}$, and $E = E_R(k)$. We will need to use the fact (see [5]) that every injective module is a direct sum of indecomposable injective modules, and each indecomposable injective module is isomorphic to $E(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R.

Lemma 2 *If N is a reflexive R-module, then the Bass numbers $\mu_i(N)$ are finite for all i.*

Proof. Recall that if $0 \to N \to E^0(N) \to E^1(N) \to \cdots$ is a minimal injective resolution of N, then $\mu_i(N)$ is the number of copies of E appearing in a direct sum decomposition of $E^i(N)$ into indecomposable injective modules. Let S be a finitely generated submodule of N with N/S artinian. Then it is well-known that $\mu_i(S) < \infty$ for all i (by [1, Lemma 2.7]), and it is easy to see that $\mu_i(A) < \infty$ for all i for any artinian module A. For in this case each $E^i(A)$ is a finite direct sum of copies of E. But we also know that $\mu_i(N)$ is the dimension of the k-vector space $\text{Ext}^i(k, N)$, i.e. $\mu_i(N) = \dim_k \text{Ext}^i(k, N)$. (See [6, Theorem 18.7], for example.) The result then follows from the long exact sequence

$$\cdots \to \text{Ext}^i(k, S) \to \text{Ext}^i(k, N) \to \text{Ext}^i(k, N/S) \to \cdots$$

and the fact that both $\text{Ext}^i(k, S)$ and $\text{Ext}^i(k, N/S)$ are finite dimensional. □

Lemma 3 *Assume that A is an artinian reflexive module and that N is reflexive. Then $Ext^i_R(A, N)$ and $Tor^R_i(A, N)$ are reflexive for all i and*

$$D(Ext^i_R(A, N)) \cong Tor^R_i(A, D(N)).$$

Proof. Let $0 \to N \to E^0(N) \to E^1(N) \to \cdots$ be a minimal injective resolution of N, and let $E^\bullet$ be the complex $0 \to E^0(N) \to E^1(N) \to \cdots$. Let $D^\bullet$ be the subcomplex of $E^\bullet$ with $D^i \subset E^i(N)$ being all x annihilated by some power of $\mathfrak{m}$. It is easy to verify that $D^\bullet$ is a subcomplex; in fact, D^i is just the sum of the E's in a decomposition of $E^i(N)$ into $E(R/\mathfrak{p})$'s for prime ideals $\mathfrak{p}$. We have an exact sequence of complexes of injective modules $0 \to D^\bullet \to E^\bullet \to E^\bullet/D^\bullet \to 0$ which splits at each level. So the sequence

$$0 \to \text{Hom}(A, D^\bullet) \to \text{Hom}(A, E^\bullet) \to \text{Hom}(A, E^\bullet/D^\bullet) \to 0$$

is exact. Since A is artinian, $\text{Hom}(A, E(R/\mathfrak{p}) = 0$ for $\mathfrak{p} \neq \mathfrak{m}$. This follows because some power of $\mathfrak{m}$ kills each element of A, yet if $x \notin \mathfrak{p}$ then x acts as an automorphism on $E(R/\mathfrak{p})$. Therefore, the complex $\text{Hom}(A, E^\bullet/D^\bullet)$ is zero and

$$H^i(\text{Hom}(A, D^\bullet)) \cong H^i(\text{Hom}(A, E^\bullet)) \cong \text{Ext}^i(A, N).$$

Since N is reflexive, by Lemma 2 each D^i is a direct sum of a finite number of copies of E. Hence each $\text{Hom}(A, D^i)$ is reflexive, since each $\text{Hom}(A, D^i)$ is a direct sum of a finite number of copies of $\text{D}(A)$ and A is assumed to be reflexive. Thus $\text{Hom}(A, D^\bullet)$ is a complex of reflexive modules, so it's homology is reflexive for each i. Therefore, $\text{Ext}^i(A, N)$ is reflexive.

Now a module is reflexive if and only if it's dual is reflexive. So $\mathrm{D}(N)$ is reflexive and the isomorphism $\mathrm{D}(\mathrm{Tor}_i(A,N)) \cong \mathrm{Ext}^i(A,\mathrm{D}(N))$ implies that $\mathrm{Tor}_i(A,N)$ is reflexive. Finally, use this isomorphism once more: we know $\mathrm{D}(\mathrm{Tor}_i(A,\mathrm{D}(N))) \cong \mathrm{Ext}^i(A,\mathrm{DD}(N)) \cong \mathrm{Ext}^i(A,N)$. So take duals and use the fact that $\mathrm{Tor}_i(A,\mathrm{D}(N))$ is reflexive to get $\mathrm{D}(\mathrm{Ext}^i(A,N)) \cong \mathrm{Tor}_i(A,\mathrm{D}(N))$. □

Proof of Theorem 3. If M is reflexive, it has a finitely generated (complete) submodule S such that M/S is artinian. Since M is reflexive, both S and M/S are reflexive. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\longrightarrow & \mathrm{D}(\mathrm{Ext}(S,N)) & \longrightarrow & \mathrm{D}(\mathrm{Ext}(M,N)) & \longrightarrow & \mathrm{D}(\mathrm{Ext}(M/S,N)) & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & \mathrm{Tor}(S,\mathrm{D}(N)) & \longrightarrow & \mathrm{Tor}(M,\mathrm{D}(N)) & \longrightarrow & \mathrm{Tor}(M/S,\mathrm{D}(N)) & \longrightarrow
\end{array}$$

The theorem follows since the two outside maps are isomorphisms of reflexive modules. □

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