A Generalized Alternating Harmonic Series

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In college calculus, it is well known that the alternating harmonic series converges to $\ln 2$, that is

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

In this note, we consider a generalized version of the alternating harmonic series. For a positive integer k, we consider the series

$$S_k = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right)$$
$$+ \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}\right) - \left(\frac{1}{3k+1} + \frac{1}{3k+2} + \cdots + \frac{1}{4k}\right) \pm \cdots$$

We will derive the analytic expression for S_k, and study its relationship to the Harmonic number

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{m=1}^{k} \frac{1}{m}.$$

1 Closed Form for S_k

By defining

$$a_n(k) = \frac{1}{(n-1)k+1} + \frac{1}{(n-1)k+2} + \cdots + \frac{1}{(n-1)k+k} = H_{nk} - H_{(n-1)k}, \qquad (1)$$

where we denote $H_0 = 0$, we have

$$S_k = \sum_{n=1}^{\infty} (-1)^{n-1} a_n(k). \qquad (2)$$

Clearly $a_n(k) > 0$ and decreases to 0 as n increases, thus S_k is well-defined.

We first note that the harmonic number has an integral expression

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \int_0^1 dx + \int_0^1 x dx + \cdots + \int_0^1 x^{n-1} dx$$
$$= \int_0^1 (1 + x + \cdots + x^{n-1}) dx = \int_0^1 \frac{1-x^n}{1-x} dx.$$

Thus, $a_n(k)$ defined in Eq. (1) is

$$a_n(k) = H_{nk} - H_{(n-1)k} = \int_0^1 \frac{1-x^{nk}}{1-x}dx - \int_0^1 \frac{1-x^{(n-1)k}}{1-x}dx$$
$$= \int_0^1 \frac{x^{(n-1)k} - x^{nk}}{1-x}dx = \int_0^1 x^{(n-1)k}\frac{1-x^k}{1-x}dx.$$

By Eq. (2),

$$S_k = \sum_{n=1}^{\infty}(-1)^{n-1}a_n(k) = \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{(n-1)k}\frac{1-x^k}{1-x}dx$$
$$= \sum_{n=1}^{\infty}\int_0^1 \left(-x^k\right)^{n-1}\frac{1-x^k}{1-x}dx = \int_0^1 \sum_{n=1}^{\infty}\left(-x^k\right)^{n-1}\frac{1-x^k}{1-x}dx$$
$$= \int_0^1 \frac{1}{1+x^k}\frac{1-x^k}{1-x}dx = \int_0^1 \frac{1+x+\cdots+x^{k-1}}{1+x^k}dx. \tag{3}$$

Now, we need the following formula

$$\int_0^1 \frac{x^{m-1}+x^{k-m-1}}{1+x^k}dx = \frac{\pi}{k\sin\frac{m\pi}{k}} \quad \text{for } k > m > 0. \tag{4}$$

Eq. (4) is on page 323, formula 3.244.1 in the book by I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Edition, Academic Press, 2007.

We can rewrite Eq. (3) as

$$S_k = \sum_{m=1}^{k-1}\int_0^1 \frac{x^{m-1}}{1+x^k}dx + \int_0^1 \frac{x^{k-1}}{1+x^k}dx = \sum_{m=1}^{k-1}\int_0^1 \frac{x^{m-1}}{1+x^k}dx + \frac{1}{k}\ln 2$$
$$= \frac{1}{2}\sum_{m=1}^{k-1}\int_0^1 \frac{x^{m-1}+x^{k-m-1}}{1+x^k}dx + \frac{1}{k}\ln 2 \tag{5}$$
$$= \sum_{m=1}^{k-1}\frac{\pi}{2k\sin\frac{m\pi}{k}} + \frac{1}{k}\ln 2. \tag{6}$$

In summary, the closed form for S_k is given as

$$S_k = \sum_{m=1}^{k-1}\frac{\pi}{2k\sin\frac{m\pi}{k}} + \frac{1}{k}\ln 2. \tag{7}$$

2 Relationship between S_k and H_k

In Eq. (3), since $1 < 1 + x^k < 2$, we have

$$S_k = \int_0^1 \frac{1+x+\cdots+x^{k-1}}{1+x^k}dx < \int_0^1 (1+x+\cdots+x^{k-1})dx = H_k$$

and

$$S_k = \int_0^1 \frac{1+x+\cdots+x^{k-1}}{1+x^k}dx > \frac{1}{2}\int_0^1 (1+x+\cdots+x^{k-1})dx = \frac{1}{2}H_k$$

for all k. This is to say that we can bound S_k by H_k as

$$\frac{1}{2}H_k < S_k < H_k.$$

By our notations in Eq. (1) and (2), we have

$$\begin{aligned} D_k = H_k - S_k = H_k - \sum_{n=1}^{\infty}(-1)^{n-1}a_n(k) &= -\sum_{n=2}^{\infty}(-1)^{n-1}a_n(k) = \sum_{n=2}^{\infty}(-1)^n a_n(k) \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}a_{n+1}(k) = \sum_{n=1}^{\infty}(-1)^{n+1}[H_{(n+1)k} - H_{nk}]. \end{aligned} \tag{8}$$

From part (a), D_k exist for every k.

If k is large, there is approximation to the harmonic number

$$H_{nk} \approx \ln nk + \gamma \quad \text{and} \quad H_{(n+1)k} \approx \ln(n+1)k + \gamma,$$

where γ is the Euler-Mascheroni constant. Therefore

$$H_{(n+1)k} - H_{nk} \approx \ln\frac{n+1}{n}.$$

Thus for large k, we have

$$D_k \approx \sum_{n=1}^{\infty}(-1)^{n+1}\ln\frac{n+1}{n} = \prod_{m=1}^{\infty}\frac{(2m)^2}{(2m-1)(2m+1)} = \ln\frac{\pi}{2},$$

where we used the Wallis formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}.$$

Therefore, we would conjecture that $\lim_{k\to\infty}(H_k - S_k) = \ln\frac{\pi}{2}$. We will prove it is indeed so. For this purpose, consider the $2m$ partial sum

$$D_{2m,k} = \sum_{n=1}^{2m}(-1)^{n+1}a_{n+1}(k) = \sum_{n=1}^{2m}(-1)^{n+1}[H_{(n+1)k} - H_{nk}].$$

From the discussion in part (a), we know that $D_{2m,k}$ is increasing in m, and the limit $D_k > D_{2m,k}$ for any m.

In the following, we will prove that for any $\epsilon > 0$, $|D_k - \ln(\pi/2)| < 3\epsilon$ for k larger than some K.

Step I: Note that

$$\begin{aligned}H_{(n+1)k} - H_{nk} &= \int_0^1 \frac{1 - x^{(n+1)k}}{1-x} dx - \int_0^1 \frac{1-x^{nk}}{1-x} dx \\ &= \int_0^1 \frac{x^{nk} - x^{(n+1)k}}{1-x} dx = \int_0^1 x^{nk} \frac{1-x^k}{1-x} dx.\end{aligned}$$

Then, the tail sum is

$$\begin{aligned}D_k - D_{2m,k} &= \sum_{n=2m+1}^{\infty} (-1)^{n+1} [H_{(n+1)k} - H_{nk}] = \sum_{n=2m+1}^{\infty} (-1)^{n+1} \int_0^1 x^{nk} \frac{1-x^k}{1-x} dx \\ &= -\sum_{n=2m+1}^{\infty} \int_0^1 \left(-x^k\right)^n \frac{1-x^k}{1-x} dx = -\int_0^1 \sum_{n=2m+1}^{\infty} \left(-x^k\right)^n \frac{1-x^k}{1-x} dx \\ &= \int_0^1 \frac{x^{k(2m+1)}}{1+x^k} \frac{1-x^k}{1-x} dx = \int_0^1 x^{k(2m+1)} \frac{1 + x + \cdots + x^{k-1}}{1+x^k} dx \\ &< \int_0^1 x^{k(2m+1)} k dx = \frac{k}{k(2m+1)+1} < \frac{1}{2m+1}.\end{aligned}$$

Thus, for any $\epsilon > 0$, there is an N_1 such that for all $n > N_1$,

$$D_k - D_{2n,k} < D_k - D_{2N_1,k} < \epsilon \tag{9}$$

for any k, where the first inequality follows from the increasing property of $D_{2m,k}$.

Step II: Recall that the product of positive real numbers $\prod_{n=1}^{\infty} a_n$ converges to a nonzero real number if and only if the sum $\sum_{n=1}^{\infty} \ln a_n$ converges. That is, for any $\epsilon > 0$, there is an N_2 such that

$$\left| \ln \prod_{n>N_2} a_n \right| = \left| \sum_{n>N_2} \ln a_n \right| < \epsilon.$$

Applying this principle to Wallis formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)},$$

we have the conclusion that, for any $\epsilon > 0$, there is an N_2, such that

$$\left| \ln \prod_{n>N_2} \frac{(2n)^2}{(2n-1)(2n+1)} \right| < \epsilon. \tag{10}$$

Step III: Fix $N = \max\{N_1, N_2\}$. We now consider the $2N$ partial sum of Eq. (8),

$$\begin{aligned} D_{2N,k} &= \sum_{n=1}^{2N} (-1)^{n+1}[H_{(n+1)k} - H_{nk}] \\ &= (H_{2k} - H_k) - (H_{3k} - H_{2k}) + (H_{4k} - H_{3k}) - (H_{5k} - H_{4k}) \\ &+ \cdots + (H_{2Nk} - H_{(2N-1)k}) - (H_{(2N+1)k} - H_{2Nk}) \\ &= \sum_{n=1}^{N} [(H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk})] \end{aligned}$$

Since $H_n - \ln n \to \gamma$ as $n \to \infty$, where γ is the Euler-Mascheroni constant, for any given ϵ, there is a K, such that for all $k > K$, (recall that N is fixed)

$$\ln k + \gamma - \epsilon/(4N) < H_k < \ln k + \gamma + \epsilon/(4N).$$

In particular, for any $n \geq 1$,

$$\ln(2n) + \ln k + \gamma - \epsilon/(4N) < H_{2nk} < \ln(2n) + \ln k + \gamma + \epsilon/(4N),$$

$$\ln(2n-1) + \ln k + \gamma - \epsilon/(4N) < H_{(2n-1)k} < \ln(2n-1) + \ln k + \gamma + \epsilon/(4N),$$

and

$$\ln(2n+1) + \ln k + \gamma - \epsilon/(4N) < H_{(2n+1)k} < \ln(2n+1) + \ln k + \gamma + \epsilon/(4N).$$

Therefore,

$$\ln \frac{2n}{2n-1} - \epsilon/(2N) < H_{2nk} - H_{(2n-1)k} < \ln \frac{2n}{2n-1} + \epsilon/(2N),$$

and

$$\ln \frac{2n+1}{2n} - \epsilon/(2N) < H_{(2n+1)k} - H_{2nk} < \ln \frac{2n+1}{2n} + \epsilon/(2N).$$

Consequently,

$$\ln \frac{(2n)^2}{(2n-1)(2n+1)} + \epsilon/N < (H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk}) < \ln \frac{(2n)^2}{(2n-1)(2n+1)} + \epsilon/N$$

that is

$$-\epsilon/N < [(H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk})] - \ln \frac{(2n)^2}{(2n-1)(2n+1)} < \epsilon/N. \tag{11}$$

Summing up Eq. (11), we have

$$-\epsilon < D_{2N,k} - \ln \prod_{n=1}^{N} \frac{(2n)^2}{(2n-1)(2n+1)} < \epsilon,$$

or

$$\left| D_{2N,k} - \ln \prod_{n=1}^{N} \frac{(2n)^2}{(2n-1)(2n+1)} \right| < \epsilon. \tag{12}$$

Finally, combining Eqs. (9), (10), (12), we have that for any $\epsilon > 0$, there is a K, such that for $k > K$,

$$\begin{aligned} \left| D_k - \ln \frac{\pi}{2} \right| &= \left| D_k - D_{2N,k} + D_{2N,k} - \ln \prod_{n=1}^{N} \frac{(2n)^2}{(2n-1)(2n+1)} - \ln \prod_{n>N} \frac{(2n)^2}{(2n-1)(2n+1)} \right| \\ &\leq |D_k - D_{2N,k}| + \left| D_{2N,k} - \ln \prod_{n=1}^{N} \frac{(2n)^2}{(2n-1)(2n+1)} \right| + \left| \ln \prod_{n>N} \frac{(2n)^2}{(2n-1)(2n+1)} \right| \\ &< 3\epsilon, \end{aligned} \tag{13}$$

which proves that

$$\lim_{k\to\infty} (H_k - S_k) = \lim_{k\to\infty} D_k = \ln \frac{\pi}{2}.$$