$X = \sigma_1 Z_1 + \mu_1 \qquad \underline{Y = \sigma_2 [\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_2}$

solve for Z_1, Z_2 $\qquad \Downarrow$

$$Z_1 = \frac{X-\mu_1}{\sigma_1} = h_1(X,Y). \qquad \frac{Y-\mu_2}{\sigma_2} = \rho Z_1 + \sqrt{1-\rho^2} Z_2$$

$$\therefore Z_2 = \left(\frac{Y-\mu_2}{\sigma_2} - \rho Z_1\right)\frac{1}{\sqrt{1-\rho^2}}$$

$$= \left[\frac{Y-\mu_2}{\sigma_2} - \rho\frac{X-\mu_1}{\sigma_1}\right] / \sqrt{1-\rho^2} = h_2(X,Y)$$

$$\frac{\partial h_1}{\partial x} = \frac{1}{\sigma_1} \qquad \frac{\partial h_1}{\partial y} = 0$$

$$\frac{\partial h_2}{\partial x} = -\rho/\sigma_1\sqrt{1-\rho^2} \qquad \frac{\partial h_2}{\partial y} = \frac{1}{\sqrt{1-\rho^2}\,\sigma_2}$$

$$\therefore J = \begin{pmatrix} 1/\sigma_1 & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}\,\sigma_1} & \frac{1}{\sqrt{1-\rho^2}\,\sigma_2} \end{pmatrix}. \qquad \det(J) = \frac{1}{\sqrt{1-\rho^2}\,\sigma_1\sigma_2}.$$

$$f_{X,Y}(x,y) = f_{Z_1,Z_2}(h_1, h_2)\,|\det(J)|$$

$$= \frac{1}{\sqrt{1-\rho^2}\,\sigma_1\sigma_2}\frac{1}{2\pi}\exp\left\{-\frac{1}{2}(h_1^2 + h_2^2)\right\}$$

compare to bivariate normal pdf

$$\frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2}\exp\left\{\frac{-1}{2}\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

We need to calculate $h_1^2 + h_2^2$

$$h_1^2+h_2^2 = \frac{(x-\mu_1)^2}{\sigma_1^2} + \left(\frac{y-\mu_2}{\sigma_2} - \rho\frac{x-\mu_1}{\sigma_1}\right)^2 \frac{1}{1-\rho^2}.$$

$$= \frac{1}{1-\rho^2}\left[(1-\rho^2)\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2} + \rho^2\frac{(x-\mu_1)^2}{\sigma_1^2}\right]$$

$$= \frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{x-\mu_1}{\sigma_1}\frac{y-\mu_2}{\sigma_2}\right].$$

Same as in 2D normal

$f(y|X=x) = \dfrac{f(x,y)}{f_X(x)}$ Note that given $X=x$. that

means $f_X(x)$ is given. also the x-related terms in $f(x,y)$ are all given. —— they are constants.

$\therefore$ $f(y|X=x) \propto f(x,y)$. $\longrightarrow$ there is only a constant multiplier which doesn't matter

$$= \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\,\frac{x-\mu_1}{\sigma_1}\,\frac{y-\mu_2}{\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

$$\boxed{\text{if } x \text{ given} \quad e^{x+y} \propto e^{2x+y}.}$$

$$\propto \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[(y-\mu_2)^2 - 2\rho\,\sigma_2\,\frac{x-\mu_1}{\sigma_1}(y-\mu_2) + \left(\rho\,\sigma_2\,\frac{x-\mu_1}{\sigma_1}\right)^2\right]\right\}$$

$$= \exp\left\{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left(y-\mu_2-\rho\,\sigma_2\,\frac{x-\mu_1}{\sigma_1}\right)^2\right]$$

which is the major part of

$$\underline{\underline{N\left(\mu_2+\rho\,\sigma_2\,\frac{x-\mu_1}{\sigma_1},\ (1-\rho^2)\sigma_2^2\right).}}$$