Math 540/640: Statistical Theory I

HW #8

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Problem 1: Assume that $Var(X) = 1$, $Var(Y) = 4$ and $Cov(X, Y) = 1$. Define $U = 2X - 3Y$ and $V = 3X + 2Y$.

(a) Use the basic properties of covariance to compute $Cov(U, V)$.

(b) Use the variance formula to compute $Var(U)$ and $Var(V)$.

(c) From (a) and (b) find $\rho(U, V)$

Problem 2: Let $X \sim N(0, 1)$ and $Y = X^2$. Compute $Cov(X, Y)$ and $Cor(X, Y)$.

Y completely depends on X. But your (correct) calculation shows $Cor(X, Y)$ is zero. So this example shows zero correlation doesn't always imply independence, although independent random variables have zero correlation (or we say they are uncorrelated.) Can you provide another example of two dependent random variables whose correlation is zero?

Problem 3: Let $X \sim Unif(0, 1)$.

(a) Compute the MGF of X.

(b) Compute the k-th moment of X by taking derivatives of the MGF, k is an arbitrary integer. We can also use $E[X_k] = \int_0^1 x^k dx$ to directly compute the k-th moment of X. Compare both results.

(c) Let $X_1, X_2, \cdots, X_n$ be n iid $Unif(0, 1)$ RVs. Compute the MGF of $\frac{X_1+\cdots+X_n}{n}$.

Problem 4: Let X be a discrete random variable with probability function as follows $f(0) = 0.5$, $f(1) = f(-1) = 0.25$.

(a) Compute the MGF of X.

(b) Let $X_1, X_2, \cdots, X_n$ be n iid RVs whose distribution is the same as that of X. Compute the MGF of $\frac{X_1+\cdots+X_n}{\sqrt{n}}$.

(c) Now suppose the MGF of a random variable Y is

$$\psi_Y(t) = \frac{e^t + e^{-t} + 2}{4}$$

Can you identify the distribution of Y?

Problem 5: (a) Let X be a random variable with the pdf $f(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$. Compute the MGF of X.

(b) Suppose X_1 and X_2 are independent $N(0, 1)$. Let $Y = X_1 + X_2$. Compute the MGF of Y.

(c) Compare your results in (a) and (b). What do you observe? Can you write down the pdf of Y?

Problem 6: Assume $X_1, \cdots, X_n$ are n iid random variables and $X_i \sim Unif(-a, a)$.

(a) Identify the constant a such that $E(X) = 0$ and $Var(X) = 1$.

(b) Let a be the number you identify in part (a). Now let $Y = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$. Compute the MGF of Y.

(c) Directly evaluate the limit of the MGF of Y by letting $n \to \infty$. By CLT what should be the limit of the MGF of Y?