Recall that in a finite sample space with the equally-likely-outcomes probability model, $P(A) = \frac{k}{n}$ where k is the number of outcomes in the event A and n is the total number of all possible outcomes. We will learn some common methods of counting to calculate k and n.

Introductory example: If 2 balls are randomly chosen from a bowl containing 6 white and 5 black balls. What is the probability that one of the drawn balls is white and the other is black?

Solution: There are 11 choices for the first drawn ball, then there are 10 choices for the second drawn ball. So there are $11 \times 10 = 110$ possible outcomes. If the first ball is white and the second is black, there are $6 \times 5 = 30$ possible outcomes. Likewise, if the first ball is black and the second is white, there are $5 \times 6 = 30$ possible outcomes. So there are $30 + 30 = 60$ outcomes in the event. Then the probability is $60/110 = 6/11$.

1. Multiplication Rule

If an experiment e is a composition of experiments $e_1, e2, \ldots, e_k$, where the sample space S_i of experiment e_i contains n_i outcomes, for each $i = 1, 2, \ldots, k$. The number of outcomes in e is the product: $n_1 \times n_2 \times \cdots n_k$.

How to use the rule? The key is to think e as a composition of simple experiments in the sense that n_i is easy to count for all $i = 1, 2. \ldots, k$.

Remark: The multiplication rule is the fundamental counting technique. We will use it to derive many other useful counting methods.

Ex1: Throw two dice (green and red). suppose we are interested in the pair of outcomes (x, y). What is the total number of outcomes?

Solution: let e_1 = throw the red die and e_2 = throw the green die. Our experiment is a composition of e_1 and e_2. $n_1 = 6$ and $n_2 = 6$, so the number=$6 \times 6 = 36$.

Ex2: How many distinct arrangements of the letters A, B, C are possible?

Solution: Let e_1 = choose a position for letter A, e_2 = choose a position for letter B after e_1 and e_3 = choose a position for letter C after e_1, e_2. Our experiment is a composition of e_1, e_2 and e_3. Now let's count n_1, n_2, n_3.

$n_1 = 3$, because we can put A in positions 1,2 and 3.

$n_2 = 2$, because after e_1, one position is taken by A, so there are 2 possible positions left for B.

$n_3 = 1$, because after e_1, e_2, the position for C is fixed, just one choice.

Then using the multiplication rule, the number is $3 \times 2 \times 1 = 6$.

To check our calculation, w also list all the possible arrangements. They are

ABC, ACB, BCA, BAC, CBA, CAB.

Ex3: A committee of chair, secretary and treasurer is selected from 20 people. How many different committee are possible?

Solution: Let e_1 be selecting the chair, e_2 be selecting the secretary after e_1 and e_3 be selecting the treasurer after e_1, e_2. Then selecting the committee is a composition of e_1, e_2 and e_3. Now let's count n_1, n_2, n_3.

$n_1 = 20$, because anyone of the 20 people could be the chair.

$n_2 = 19$, because after e_1, one person is selected as the chair, so there are $20 - 1 = 19$ candidates for the secretary.

$n_3 = 18$, because after e_1, e_2, one person is selected as the chair and another is the secretary, so there are $20 - 2 = 18$ candidates for the treasurer.

Then using the multiplication rule, the number is $20 \times 19 \times 18 = 6840$.

Ex4: We have k balls and n baskets. $n > k$. We randomly put each ball in one of the baskets. Balls are allowed to share a basket. Find the probability that no basket has more than one ball.

Solution: First the size of the sample space is n^k, because the first ball can be in any one of the n baskets and the same is true for other balls. Let $A =$ the event that no basket has more than one ball. This is an equally likely outcomes model, so $P(A) = \frac{|A|}{n^k}$, $|A|$ is the number of outcomes in A.

If A happens, then the first ball can be in any one of the n baskets; the second ball must be in one of the rest $n-1$ baskets; the third ball must be in one of the rest $n-2$ baskets, and so on. The kth ball must be in one of the $n-k+1$ baskets. So the number of outcomes in A is $n \times (n-1) \times \cdots \times (n-k+1)$. Then $P(A) = \frac{n\times(n-1)\times\cdots\times(n-k+1)}{n^k} = (1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})$.

Ex5: The Birthday Problem There are 62 students are in a classroom. I bet a dollar that there must have two students in this classroom who have the same birthday. What is the chance that I win a dollar?

I claim that the chance is about 99.5%.

How do I get this number? It is an application of Ex4. Consider 62 students are 62 "balls" that are randomly assigned to 365 days ("baskets"). What is the complement of the event that at least two students have the same birthday? It is the event that no one day ("basket") has two students ("balls"). By Ex4 we know its probability is $(1-\frac{1}{365})(1-\frac{2}{365})\cdots(1-\frac{62-1}{365}) = 0.005$ (note $n = 365$ and $k = 62$). Then using the formula $P(A) = 1 - P(A^c)$ we see the desired probability is $1-0.005 = 0.995$.

2. Permutations

Fact 1: The number of distinct arrangements of n objects in a sequence is denoted by $P_{n,n}$ and

$$P_{n,n} = n \times (n-1) \times (n-2) \cdots \times 2 \times 1.$$

We also write the right hand side as $n!$.

Remark 1: Consider fact 1 as a generalized version of Ex2 on page 4 in which there are $n = 3$ objects. By the fact, the number in Ex2 is $P_{3,3} = 3! = 3 \times 2 \times 1 = 6$ which agrees with our previous calculation.

Fact 2: The number of distinct arrangements of k objects from n available objects in a sequence is denoted by $P_{n,k}$ and

$$P_{n,k} = n \times (n-1) \times (n-2) \cdots \times (n-k+1).$$

where $1 \leq k \leq n$.

Remark 2: Fact 2 is a further generalization of fact 1. Note that $P_{n,k} = \frac{n!}{(n-k)!}$.

Explanation of Facts 1 & 2

We only derive fact 2 since it covers fact 1 when $k = n$. The derivation is very similar to the solution of Ex2.

Consider k sequential experiments. $e_1 =$ select an object to fill the first position, e_2 =select an object to fill the second position and so on. e_k=select an object to fill the kth position. The experiment is a composition of $e_1, e_2, \ldots, e_k$.

$n_1 = n$ because anyone of the 20 people could be the chair.

$n_2 = n - 1$, because after e_1, there are $n - 1$ candidates for the second position.

Likewise, $n_k = n - k + 1$.

Then using the multiplication rule, the number is $n \times (n - 1) \times \cdots \times (n - k + 1)$.

3. Combinations

Suppose there is a set of n objects from which we choose k objects to form a subset. We want to determine the number of different subsets that can be chosen. We use $C_{n,k}$ to denote that number.

Note that we don't care about the sequence or order in the subset. No two subsets will consist of exactly the same elements.

Fact 3: $C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{k!(n-k)!}$.

For example, $C_{6,3} = \frac{6!}{3!(6-3)!} = \frac{1\times2\times3\times4\times5\times6}{(1\times2\times3)^2} = 20$.

Note that we also write $C_{n,k} = \binom{n}{k}$.

Derive Fact 3

Given a combination, we can permute the k elements to get a sequence. There are $k!$ ways to arrange the k elements. Since there are $C_{n,k}$ combinations, we know there are $C_{n,k} \times k!$ ways to form a sequence of k elements. But Fact 2 tells us that there are $P_{n,k}$ such sequences. **We count the same number using two methods, they should agree with each other.** Thus we have an equation

$$C_{n,k} \times k! = P_{n,k}$$

So $C_{n,k} = \frac{P_{n,k}}{k!}$. Note $P_{n,k} = \frac{n!}{(n-k)!}$, then we have $C_{n,k} = \frac{n!}{k!(n-k)!}$.

Ex1: Suppose that a committee composed of 8 people is to be selected from a group of 20 people. The number of different groups of people that might be on the committee $C_{20,8} = \frac{20!}{8!(20-8)!} = \frac{20!}{8!12!} = 125970$.

Ex2: How many committees with 2 Republicans, 2 Democrats and 3 Independents can be formed from a group of 6 Republican, 5 Democrat and 4 Independent senators?

Solution:

$e_1 =$ select 2 Rs from 6 R candidates, $n_1 = \binom{6}{2}$

$e_2 =$ select 2 Ds from 5 D candidates, $n_2 = \binom{5}{2}$

$e_3 =$ select 3 INDs from 4 IND candidates, $n_3 = \binom{4}{3}$

The committee selection is a composition of e_1, e_2, e_3. So the number is $n_1 \times n_2 \times n_3 = \binom{6}{2} \times \binom{5}{2} \times \binom{4}{3} = 600$.

Ex3: Suppose a fair coin is tossed 10 times. Compute the probability of (a) obtaining exactly 3 heads; (b) obtaining 3 or fewer heads.

Solution: The number of possible outcome in each tossing is 2, so the size of the sample space is 2^{10}.. "A fair coin" means the probability model is an equally-likely-outcomes model.

(a) Let $A_3 =$ the event that we have exactly 3 heads. We select 3 out of 10 tosses to let their outcomes be "Head". So the size of A_3 is $C_{10,3}$ or $\binom{10}{3}$. So $P(A_3) = \frac{\binom{10}{3}}{2^{10}} = 0.1172$.

(b) Let B be the event that we have 3 or fewer heads. Also let A_2 be the event that we have exactly 2 heads, A_1 be the event that we have exactly 1 head and A_0 be the event that we have no head. Then

$$B = A_3 \cup A_2 \cup A_1 \cup A_0.$$

From (a) we know $P(A_3) = \frac{\binom{10}{3}}{2^{10}}$. By the same arguments in (a) we find $P(A_2) = \frac{\binom{10}{2}}{2^{10}}$, $P(A_1) = \frac{\binom{10}{1}}{2^{10}}$ and $P(A_0) = \frac{\binom{10}{0}}{2^{10}}$ thus

$$P(B) = \frac{\binom{10}{3}}{2^{10}} + \frac{\binom{10}{2}}{2^{10}} + \frac{\binom{10}{1}}{2^{10}} + \frac{\binom{10}{0}}{2^{10}} = 0.1719.$$

4. The Partition Formula

Motivation example: Suppose we want to divide 12 Bridge players into 3 rooms to play the game, what is the number of all possible arrangements?

Solution: To arrange room 1, we need to choose 4 players from 12 players. So there are $\binom{12}{4}$ ways. Then we have 8 players left. To arrange room 2, we need to choose 4 players from 8 players. So there are $\binom{8}{4}$ ways. The rest 4 players are put in room 3. By the multiplication rule, we desired number is $\binom{12}{4} \times \binom{8}{4} \times 1 = 34650$.

This is an example of the partition problem.

4. The Partition Formula, contd'

In general, suppose we need to divide (partition) n objects into r groups. Let n_j be the size of group j, $j = 1, 2, \ldots, r$ such that $n_1 + n_2 + \cdots + n_r = n$. We denote by $\binom{n}{n_1, n_2, \ldots, n_r}$ the number of possible partitions.

Fact 4:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}.$$

To illustrate the partition formula, let's go back to the Bridge players example in which $n = 12$ and $n_1 = n_2 = n_3 = 4$. So the number is $\binom{12}{4,4,4} = \frac{12!}{4!4!4!} = 34650$.

Derive Fact 4

The derivation is basically given in the Bridge players example. Sequentially, we let e_j be the experiment to form group j, $1 \leq j \leq r$. Then the partition is a composition of experiments $e_1, \ldots, e_r$. There are $\binom{n}{n_1}$ ways to form group 1. Then from $n-n_1$ objects we choose n_2 objects to form group 2, there are $\binom{n-n_1}{n_2}$ ways. Keep using similar arguments, there are $n-n_1-n_2-\cdots-n_{r-2}$ objects left to form group $r-1$. After group $r-1$, there are $n-n_1-n_2-\cdots-n_{r-2}-n_{r-1}=n_r$ objects left and there is just one way to form group r. So the number is $\binom{n}{n_1}\binom{n-n_1}{n_2}\times\binom{n-n_1-n_2}{n_3}\times\cdots\times\binom{n-n_1-n_2-\cdots-n_{r-2}}{n_{r-1}}\times 1$. Using the equation $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ we can simply the expression and get the formula in Fact 4.

Example: How many distinct arrangements can be formed from the letters "m,i,s,s,i,s,s,i,p,p,i"?

Solution: We have 11 letters. Each arrangement is a partition: we need to find 1 position for "M", 4 positions for "I", 4 positions for "S", 2 positions for "P".

Using the partition formula, the number is

$$\binom{11}{1,4,4,2} = \frac{11!}{1!4!4!2!} = 34650.$$

5. Binomial and Multinomial Coefficients

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Because of the theorem, $\binom{n}{k}$ is also called binomial coefficients.

For example,

$$\begin{aligned}(x+y)^2 &= \binom{2}{0} x^0 y^{2-0} + \binom{2}{1} x^1 y^{2-1} + \binom{2}{2} x^2 y^{2-2} \\ &= y^2 + 2xy + x^2\end{aligned}$$

A special case: Let $y = 1 - x$,

$$1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

This identity is related to the binomial distribution.

Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_k)^n$$
$$=$$
$$\sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1,n_2,\ldots,n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

$\binom{n}{n_1,n_2,\ldots,n_k}$ is called multinomial coefficients.

A special case: Let $x_1 + x_2 + \ldots + x_k = 1$, then

$$1 = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

This identity is is related to the multinomial distribution.

Example: What is the coefficient of x^2y^3 in the expansion of $(1 + x + y)^{100}$?

Solution: Consider $x_1 = 1$, $x_2 = 2$, $x_3 = y$. $n = 100$, $n_2 = 2$, $n_3 = 3$. So $n_1 = n - n_2 - n_3 = 100 - 2 - 3 = 95$. Then by the multinomial theorem the coefficient is $\binom{100}{95,2,3} = 485100$.