Conditional Probability: Consider two events A and B. Their probabilities are $P(A)$ and $P(B)$. Now assume that we know B has occurred, then what is the chance that A will occur? Is the probability still P(A) if we know the fact that B occurred?

Motivation example 1: Roll a die. Let A denote the event that we get 1 and B be the event that we get an odd number. Assume it is true that we get an odd number (B occurred), what is the probability that the number is 1 (A occurs)?

Solution: $B = \{1, 3, 5\}$. If B occurred then the number must be 1, or 3 or 5. They have equally likely chances, so we think the probability of getting 1 is 1/3.

We call 1/3 the conditional probability of A given B.

A mini "Deal or No Deal" game

The game host puts \$10K, \$100 and \$1 in three briefcases. You, the player, can select any case you like. Then you are offered the second chance to select another case and the case will be opened. Suppose you see the second case does not have \$10K, then what is the chance that the first chosen case has \$10K?

Let A be the event that the first chosen case has the 10 thousand dollars. Before the second briefcase, we know $P(A) = \frac{1}{3}$. Let B be the event that the second chosen case does not contain 10k dollars. We will learn how to update the probability of A given the fact that the event B has occurred.

Definition: Consider two events A and B. The conditional probability of A given B is defined as

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

If $P(B) = 0$ then the conditional probability is not defined.

The conditional probability $P(A|B)$ describes the chance that A occurs given the knowledge that B has occurred.

Note that AB is the event that both A and B occur.

Question: What is $P(B|A)$?

By definition, $P(B|A) = \frac{P(BA)}{P(A)}$. $P(B|A)$ describes the chance that B occurs given the knowledge that A has occurred.

Ex2: A black box contains 10 white, 5 yellow and 10 black marbles. A marble is chosen at random and it is noted that it is not one of the black marbles. Then what is the probability that the chosen one is yellow?

Solution: Let A be the event that the selected marble is yellow, and let B denote that the event that the chosen one is not black. Then we need to compute $P(A|B)$. This is an equally-likely-outcomes model, so $P(B) = \frac{10+5}{10+5+10} = 0.6$. Also we see $AB = A$, so $P(AB) = P(A) = \frac{5}{10+5+10} = 0.2$.

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.2}{0.6} = 1/3.$$

Ex3: The organization for which Ms J. works for is running a dinner party for those employers having at least one son. If Ms J. is known to have two children. What is the probability that she has two sons, if she is invited to the dinner party?

Solution: Ms J. has two children. There are 4 possible combinations: (B,B), (B,G), (G,B), (G,G). Each has 1/4 probability to occur. Let A be the event that she has two sons, So $A = \{(B, B)\}$. Let B be the event that she is invited. Then $B = \{(B, B), (B, G), (G, B)\}$. $P(B) = \frac{3}{4}$. Also we note $AB = A$, so $P(AB) = P(A) = \frac{1}{4}$. Thus

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = 1/3.$$

Some may think that the probability of two boys given at least one boy is $\frac{1}{2}$, as opposed to the correct $\frac{1}{3}$, since they reason that the other child is equally likely to be a boy or a girl. Their mistake is to assume the equally likely probability. Note that initially there are 4 equally likely outcomes. Knowing the information that at least one child is boy is equivalent to knowing the outcome is not (G, G). Hence we have three equally likely outcomes (B,B), (B,G), (G,B) and the other child is twice likely to be a girl as to be a boy.

Attention: $P(A|B) \neq \frac{P(A)}{P(B)}$. Here is a counter-example! **Ex4:** Roll a die. A=observe a prime number, B=observe an even number. What is $P(A|B)$? Solution: $A = \{2, 3, 5\}$ $B = \{2, 4, 6\}$, $AB = \{2\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/6}{3/6} = 1/3.$$

$$But \ \frac{P(A)}{P(B)} = \frac{3/6}{3/6} = 1.$$

Multiplication Formula

From the equation

$$P(A|B) = \frac{P(AB)}{P(B)}$$

we obtain another identity

$$P(AB) = P(B)P(A|B) \quad (*).$$

(*) is called the multiplication formula. It is often used to compute $P(AB)$.

Ex5: Suppose a basket has 7 black balls and 5 white balls. We randomly select 2 balls from the basket. What is the probability that both balls are black? **Solution:** Let B=the first drawn ball is black, A= the second drawn ball is black. Then we need to find $P(AB)$. We will use the multiplication formula. $P(B) = \frac{7}{7+5} = \frac{7}{12}$. Given B, $P(A|B) = \frac{6}{11}$, because there are 6 black balls and 5 white balls.

$$P(AB) = P(B)P(A|B) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}.$$

Multiplication Formula, contd'

Remark 1: Note that $AB = BA = A \cap B$, thus we can also write

$$P(AB) = P(A)P(B|A) \quad (**)$$

Therefore

$$P(AB) = P(B)P(A|B) = P(A)P(B|A) \quad (***)$$

(***) is related to Bayes Theorem, the most important topic in chap 2.

Remark 2: What if we have k events $A_1, \ldots, A_k$, $k \geq 3$, Is there a similar multiplication formula for $P(A_1 A_2 \cdots A_k)$? Yes!

More than two events

$$P(A_1 A_2 \cdots A_k)$$
$$= P(A_1)P(A_2|A_1) \cdots P(A_k|A_{k-1}A_{k-2} \ldots A_1).$$

Proof: Note that $A_1 A_2 \cdots A_k = A_1 \cap A_2 \cdots \cap A_k$. In the notation $P(A_k|A_{k-1}A_{k-2} \ldots A_1)$, we think $A_{k-1}A_{k-2} \ldots A_1$ as a single event, then $P(A_k|A_{k-1}A_{k-2} \ldots A_1)$ is just the usual conditional probability of two events.

$$P(A_k|A_{k-1}A_{k-2} \ldots A_1) = \frac{P(A_k A_{k-1} A_{k-2} \ldots A_1)}{P(A_{k-1}A_{k-2} \ldots A_1)}$$

$$P(A_1)P(A_2|A_1) \cdots P(A_k|A_{k-1}A_{k-2} \ldots A_1)$$
$$= P(A_1)\frac{P(A_2A_1)}{P(A_1)}\frac{P(A_3A_2A_1)}{P(A_2A_1)} \cdots \frac{P(A_kA_{k-1}A_{k-2} \ldots A_1)}{P(A_{k-1}A_{k-2} \ldots A_1)}$$
$$= P(A_kA_{k-1}A_{k-2} \ldots A_1)$$
$$= P(A_1 A_2 \cdots A_k).$$

Ex6: Select 4 balls one at a time from a box containing r red balls and b blue balls. For example, r=10 and b=20. What is the probability of obtaining the sequence of outcomes (red, blue, red, blue)?

Solution: Let R_j be the event that the jth ball is red, $j = 1, 2, 3, 4$. Let B_j be the event that the jth ball is blue, $j = 1, 2, 3, 4$. We need to compute $P(R_1B_2R_3B_4)$. Using the multiplication formula, we have

$$P(R_1B_2R_3B_4)$$
$$= P(R_1)P(B_2|R_1)P(R_3|B_2R_1)P(B_4|R_3B_2R_1)$$
$$= \frac{r}{r+b} \times \frac{b}{r-1+b} \times \frac{r-1}{r-1+b-1} \times \frac{b-1}{r-2+b-1}$$
$$= \frac{rb(r-1)(b-1)}{(r+b)(r+b-1)(r+b-2)(r+b-3)}$$

For $r = 10, b = 20$, the probability is

$$\frac{10 \times 20 \times 9 \times 19}{30 \times 29 \times 28 \times 27} = 0.052.$$

Independent Events

Intuitively, if learning that event B has occurred does not change the probability of event A, then we say that A and B are independent. Mathematically we have the following definition.

Definition: Two events A and B are said to be *independent* if

$$P(AB) = P(A)P(B).$$

Two events A and B that are not independent are said to be *dependent*.

Note that if A and B are independent then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

So the definition agrees with our intuition.

Ex1: Rolling a die. Let $A = \{2, 4, 6\}$. Let $B = \{1, 2, 3, 4\}$. We claim A and B are independent.

Solution: $AB = \{2, 4\}$, so $P(AB) = \frac{1}{6}$. $P(A) = \frac{3}{6} = \frac{1}{2}$ and $P(B) = \frac{4}{6} = \frac{2}{3}$. $P(A)P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Therefore we find $P(AB) = P(A)P(B)$. By definition, A and B are independent.

In this example, it is hard to intuitively justify the independence between A and B. Sometimes we know the independence by physical laws or common senses. For example, Let A denote the event that the next president is female and let B denote the event that there will be a major earthquake within the next year in Nevada. We can safely assume A and B are independent.

Suppose we know A and B are independent and we know B didn't occur. Will this information change the probability of A? In another word, are A and B^c independent or dependent?

Fact: If A and B are independent, so are A and B^c.

Proof: We need to show $P(AB^c) = P(A)P(B^c)$. Note that $P(AB^c) + P(AB) = P(A)$, because $AB^c \cup AB = A$ and $AB^c \cap AB = \emptyset$. Thus $P(AB^c) = P(A) - P(AB)$. But $P(AB) = P(A)P(B)$, since A and B are independent. So

$$P(AB^c) = P(A) - P(A)P(B) = P(A)(1 - P(B))$$

Using $P(B^c) = 1 - P(B)$ we prove $P(AB^c) = P(A)P(B^c)$.

Question: If A and B are independent, How about A^c and B, or A^c and B^c? **Answer:** they are independent! I leave the proof to you as an exercise (hint: repeatedly use the above fact).

Ex2: Roll 2 dice. Let A denote the event that the sum of the dices is 6. Let B denote the event that the first die is 4. We claim A and B are dependent.

$A = \{(5,1),(1,5),(4,2),(2,4),(3,3)\}$

$$P(A) = \frac{5}{36}$$

$B = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

$$P(B) = \frac{6}{36} = \frac{1}{6}.$$

$AB = \{(4,2)\}$. $P(AB) = \frac{1}{36}$.

$P(AB) < P(A)P(B)$ so A and B are dependent.

n **Independent Events,** $n \geq 3$

Definition: The events $A_1, A_2, \ldots, A_n$ are said to be independent if, for every subset $A_{1'}, A_{2'}, \ldots, A_{r'}$ $r \leq n$ of these events

$$P(A_{1'}A_{2'} \cdots A_{r'}) = P(A_{1'})P(A_{2'}) \cdots P(A_{r'})$$

We define an infinite set of events to be independent if every finite subset of these events is independent.

Ex3: A machine produces a defective item with probability p and produces a non-defective item with probability $q = 1 - p$. Six items are selected at random and inspected. The outcomes for these six items are independent. Compute the probability that exactly two of the six items are defective.

Solution: Let D_j denote the event that the jth item is defective. Let N_j denote the event that the jth item is non-defective. Let A denote the event that there are 2 defective items and 4 non-defective items. For example, one such outcome can be $N_1D_2N_3N_4D_5N_6$. By independence,

$$
\begin{aligned}
&P(N_1D_2N_3N_4D_5N_6) \\
&= P(N_1)P(D_2)P(N_3)P(N_4)P(D_5)P(N_6) \\
&= qpqqpq = p^2q^4
\end{aligned}
$$

There are $\binom{6}{2}$ such outcomes and each has the same probability. Thus the desired probability is $\binom{6}{2}p^2q^4 = 15p^2q^4$.

Bayes Theorem

Partition of the sample space: Disjoint events $A_1, \ldots, A_n$ are said to be a partition of the sample space S if

$$S = A_1 \cup A_2 \cup \cdots \cup A_n.$$

An obvious partition: For any event B, $B \cup B^c = S$, B and B^c are disjoint. So B and B^c are a partition of S.

Total probability formula: Given a partition $A_1, \ldots, A_n$, for any event B, we have

$$P(B) = \sum_{i=1}^{n} P(BA_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Proof: $BA_1, \ldots, BA_n$ are disjoint. $\cup_{i=1}^{n}(B \cap A_i) = B \cap (\cup_{i=1}^{n} A_i) = B \cap S = B$ because $\{A_i\}_{i=1}^{n}$ is a partition of S. Therefore

$$P(B) = P(\cup_{i=1}^{n} BA_i) = \sum_{i=1}^{n} P(BA_i),$$

but we know $P(BA_i) = P(B|A_i)P(A_i)$.

Bayes Theorem: Consider a partition $\{B_i\}_{i=1}^n$. For each i, we have

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^n P(A|B_i)P(B_i)}.$$

Special case: Consider two events A and B.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Proof:

$$P(B|A) = \frac{P(BA)}{P(A)} \quad definition$$

$$P(BA) = P(A|B)P(B) \quad multiplication\ formula$$

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

[total probability formula]. Combining the three equation we prove the special case. Similar arguments prove the general case.

Ex 1: A blood test is 95% effective in detecting a certain disease when it is present. The test also yields a false positive result for 1% of the healthy people tested. If 0.5% of the population actually have the disease, what is the probability that a person has the disease when his test result is positive?

Solution: Let D=has the disease and E=positive test result. We wan to find $P(D|E)$. Note that $P(E|D) = 0.95$ and $P(E|D^c) = 0.01$ (the false positive rate) $P(D) = 0.005$ (0.5% of the population actually have the disease) $P(D^c) = 1 - P(D) = 0.995$.

Using Bayes theorem,

$$\begin{aligned} P(D|E) &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} \\ &= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} \\ &= 0.323 \end{aligned}$$

So about 32% of the people whose test results are positive actually have the disease.

Ex 2: A plane is missing and it is presumed that it was equally likely to have gone down in any of 2 regions. Let $1 - \alpha_i$ be the probability that the plane will be found if the plane is in region i, $i = 1, 2$. So α_i is the overlook probability. For instance, $\alpha_1 = 0.1$, $\alpha_2 = 0.05$. What is the probability that the plane is in region 2 given that a search in region 1 is unsuccessful?

Solution: Let E be the event that the search is unsuccessful in region 1. Let R_i denote the event that the plane is in region i, $i = 1, 2$. We need to find $P(R_2|E)$.

Using Bayes theorem,

$$
\begin{aligned}
P(R_2|E) &= \frac{P(E|R_2)P(R_2)}{P(E|R_1)P(R_1) + P(E|R_2)P(R_2)} \\
&= \frac{1 \times \frac{1}{2}}{\alpha_1 \frac{1}{2} + 1 \times \frac{1}{2}} \\
&= \frac{1}{1 + \alpha_1} = \frac{1}{1 + 0.1} = \frac{10}{11} = 0.91.
\end{aligned}
$$

Ex 3: Three cards are placed in a black box. One is red on both sides, the second is blue one both sides and the third one is red on one side and blue on the other side. One card is selected at random and placed on the table. Suppose that the color showing on the card is red. What is the probability that the color underneath is also red?

Solution: Let A be the event that the bottom side of the selected card is red and B be the event that the top side of the selected card is red. We need to find $P(A|B)$.

C_1 =the selected card is red/red; C_2 = the selected card is blue/blue; C_3=the selected card is red/blue. Using Bayes theorem,

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{P(C_1)}{P(B|C_1)P(C_1)+P(B|C_2)P(C_2)+P(B|C_3)P(C_3)}$$

$$= \frac{1/3}{1\times 1/3+0\times 1/3+1/2\times 1/3}$$

$$= \frac{2}{3}.$$

Solution to "Deal or No Deal" Problem on page 2

We use Bayes theorem to update the probability that the first case has 10k dollars given that the second chosen case does not have 10k dollars. A=the first chosen case has the 10K dollars. B=the second chosen case does not contain 10K dollars.

$P(A) = \frac{1}{3}$ and $P(A^c) = \frac{2}{3}$. $P(B|A) = 1$ and $P(B|A^c) = \frac{1}{2}$. Using Bayes theorem,

$$\begin{aligned} P(A|B) &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \\ &= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 1/2 \times \frac{2}{3}} \\ &= \frac{1}{2}. \end{aligned}$$