Marginal Distribution and Marginal Density: (X, Y) has the joint pdf $f(x, y)$. The marginal density functions of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx.$$

Explanation: We can actually derive the above equations. Take an arbitrary a and consider the region $A = \{(x, y) : x \leq a\}$.

$P(A) = P(X \leq a) = F_X(a)$. But we know

$$P(A) = \iint_A f(x, y)dxdy = \int_{-\infty}^{a} dx[\int_{-\infty}^{\infty} dyf(x, y)]$$

Let $g(x) = \int_{-\infty}^{\infty} dyf(x, y)$. Then

$$F_X(a) = \int_{-\infty}^{a} dxg(x) \quad \text{for all possible } a$$

which implies $f_X(x) = g(x) = \int_{-\infty}^{\infty} dyf(x, y)$. Similarly, we prove $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$.

Ex.1

Suppose the joint pdf is given by

$$f(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

We compute the marginal density $f_Y(y)$. If $|y| > 1$ then $f(x,y) = 0$ for all x

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} 0dx = 0.$$

If $|y| \leq 1$ then $f(x,y) = 0$ for $|x| > \sqrt{1-y^2}$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx = \frac{2\sqrt{1-y^2}}{\pi}.$$

So

$$f_Y(y) = \begin{cases} \frac{2\sqrt{1-y^2}}{\pi} & \text{if } -1 \leq y \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$Similarly,\ f_X(x) = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & \text{if } -1 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Note the marginal is not uniform!

Indicator Functions

$$I(\text{argument}) = \begin{cases} 1 & \text{if argument is true} \\ 0 & \text{if argument is false} \end{cases}$$

For example, in Ex.1

$$f(x,y) = \frac{1}{\pi} I(x^2 + y^2 \le 1)$$

$$f_X(x) = \frac{2\sqrt{1-x^2}}{\pi} I(-1 \le x \le 1)$$

$$f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi} I(-1 \le y \le 1)$$

Note that

$$I(0 < x < 1, 0 < y < 1) = I(0 < x < 1) I(0 < y < 1)$$

and in general,

$$\begin{aligned} &I(\text{argument 1}) I(\text{argument 2}) \\ &\quad = I(\text{arguments 1 and 2}) \end{aligned}$$

Ex.2

Suppose the joint pdf is given by

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}.$$

We compute the marginal density $f_X(x)$.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times 1 \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
\end{aligned}$$

Likewise, $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$. So X and Y are $N(0,1)$.

Independent Random Variables

Definition: We say X and Y are independent, if for every two sets A and B of real numbers,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

Remark 1: Suppose (X, Y) has a continuous joint pdf $f(x, y)$. Suppose the marginal densities of X and Y are $f_X(x)$ and $f_Y(y)$. Then X and Y are independent if and only if

$$f(x, y) = f_X(x) f_Y(y) \quad \underline{\text{for all pairs of (x, y)}}.$$

Remark 2: Suppose (X, Y) has a discrete joint pf $f(x, y)$. Then X and Y are independent if and only if

$$f(x, y) = f_X(x) f_Y(y) \quad \underline{\text{for all pairs of (x, y)}}.$$

where $f_X(x)$ and $f_Y(y)$ are the probability function (p.f.) of X and Y.

In Ex.1, we claim X and Y are not independent, because

$$f_X(\sqrt{\frac{2}{3}})f_Y(\sqrt{\frac{2}{3}}) = \frac{4\sqrt{1/3}\sqrt{1/3}}{\pi^2} = \frac{4}{3\pi^2}$$

$$But\ \ f(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}) = 0$$

In Ex.2, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

$$f_X(x)f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

Therefore we checked for every pair (x, y)

$$f_X(x)f_Y(y) = f(x, y)$$

which means X and Y are independent.

Factorization Theorem

If we can find two univariate functions $h(x)$ and $q(y)$ such that

$$(*) \quad f(x,y) = g_1(x)g_2(y),$$

then X and Y are independent.

Note that if X and Y are independent, then $f(x,y) = f_X(x)f_Y(y)$. So the condition (*) does hold, because we let $g_1(x) = f_X(x)$ and $g_2(y) = f_Y(y)$.

The nice thing about Factorization theorem is that often we can find such a factorization of $f(x,y)$ as in (*) without computing the marginal densities, because $g_1(x)$ is not necessarily $f_X(x)$ and $g_2(y)$ is not necessarily $f_Y(y)$.

Ex.3

Suppose the joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} 2e^{-x-2y} & \text{if } 0 < x, 0 < y \\ 0 & \text{Otherwise} \end{cases}$$

Using indicator function, we can write

$$f(x,y) = 2e^{-x-2y} I(x > 0, y > 0).$$

We choose $g_1(x) = 2e^{-x} I(x > 0)$ and $g_2(y) = e^{-2y} I(y > 0)$. Check

$$\begin{aligned} g_1(x) g_2(y) &= 2e^{-x} I(x > 0) e^{-2y} I(y > 0) \\ &= 2e^{-x-2y} I(x > o) I(y > 0) \\ &= 2e^{-x-2y} I(x > 0, y > 0) \\ &= f(x,y) \end{aligned}$$

By Factorization theorem, we know X and Y are independent.

Proof of Factorization Theorem

Suppose condition (*) holds for some $g_1(x)$ and $g_2(y)$.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\infty}^{\infty} g_1(x)g_2(y)dy = g_1(x)(\int_{-\infty}^{\infty} g_2(y)dy) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \int_{-\infty}^{\infty} g_1(x)g_2(y)dx = g_2(y)(\int_{-\infty}^{\infty} g_1(x)dx) \end{aligned}$$

From $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy$ we also find

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x)g_2(y)dxdy \\ &= (\int_{-\infty}^{\infty} g_1(x)dx)(\int_{-\infty}^{\infty} g_2(y)dy). \end{aligned}$$

Thus we have

$$\begin{aligned} f_X(x)f_Y(y) &= g_1(x)g_2(y)(\int_{-\infty}^{\infty} g_2(y)dy)(\int_{-\infty}^{\infty} g_1(x)dx) \\ &= g_1(x)g_2(y) = f(x,y). \end{aligned}$$

So by definition X and Y are independent.

Conditional pdf

Suppose (X, Y) has a joint pdf $f(x, y)$. The marginal pdf's of X and Y are denoted by $f_X(x)$ and $f_Y(y)$.

Take any x, if $f_X(x) > 0$, then we say the conditional pdf of Y given $X = x$ is

$$f_Y(y|X = x) = \frac{f(x, y)}{f_X(x)}.$$

$f_Y(y|X = x)$ describes the distribution of Y when we observe that the value of X is x.

Take any y, if $f_Y(y) > 0$, then we say the conditional pdf of X given $Y = y$ is

$$f_X(x|Y = y) = \frac{f(x, y)}{f_Y(y)}.$$

$f_X(x|Y = y)$ describes the distribution of X when we observe that the value of Y is y.

For any two sets A and B, $P(X \in A|Y = y)$ means the probability that the value of X is in A when we observe that the value of Y is y. $P(Y \in B|X = x)$ means the probability that the value of Y is in B when we observe that the value of X is x.

We can compute them by

$$P(X \in A|Y = y) = \int_A f_X(x|Y = y)dx$$

and

$$P(Y \in B|X = x) = \int_B f_Y(y|X = x)dy$$

Important identities:

$$f_X(x)f_Y(y|X = x) = f(x, y).$$

$$f_Y(y)f_X(x|Y = y) = f(x, y).$$

Ex.1 Independent X and Y

If X and Y are independent, then $f(x,y) = f_X(x)f_Y(y)$.

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)} = f_Y(y)$$

$$f_X(x|Y=y) = \frac{f(x,y)}{f_Y(y)} = f_X(x)$$

Therefore, the conditional pdf is exactly the marginal pdf.

The following statements are equivalent:

1. X and Y are independent

2. $f(x,y) = f_X(x)f_Y(y)$ for all (x,y)

3. $f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)} = f_Y(y)$ for all (x,y)

4. $f_X(x|Y=y) = \frac{f(x,y)}{f_Y(y)} = f_X(x)$ for all (x,y)

Ex.2

Suppose the joint pdf

$$f(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Find $f_X(x|Y=0.5)$.

Solution: We have calculated

$$f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}\,I(-1 \le y \le 1)$$

$$\begin{aligned} f_X(x|Y=0.5) &= \frac{f(x,0.5)}{f_Y(0.5)} = \frac{\frac{1}{\pi}I(x^2+0.5^2 \le 1)}{\frac{2}{\pi}\sqrt{1-0.5^2}} \\ &= \frac{1}{\sqrt{3}}I(|x| \le \sqrt{0.75}). \end{aligned}$$

Ex.3

Suppose the joint pdf is given by

$$f(x,y) = \begin{cases} 6(x-y) & \text{if } 0 < y < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

Compute $f_Y(y|X = 0.6)$.

Solution: First compute $f_X(x = 0.6)$.

$$\begin{aligned} f_X(0.6) &= \int_{-\infty}^{\infty} f(0.6, y)dy \\ &= \int_0^{0.6} 6(0.6 - y)dy = 1.08 \end{aligned}$$

Then by definition of conditional pdf,

$$\begin{aligned} f_Y(y|X = 0.6) &= \frac{f(0.6, y)}{f_X(0.6)} \\ &= \frac{6(0.6 - y)I(0 < y < 0.6)}{1.08} \\ &= \frac{(0.6 - y)}{0.18} I(0 < y < 0.6). \end{aligned}$$

Continuous Bayes Theorem

Question: suppose we know the marginal density of X $f_X(x)$ and the conditional pdf of Y given X: $f_Y(y|X = x)$. How to get the conditional pdf of X given Y? For example, what is $f_X(x|Y = 1)$?

Solution: Let $f(x, y)$ be the joint pdf of X and Y, then by definition

$$f_X(x|Y = 1) = \frac{f(x, 1)}{f_Y(1)}.$$

So we need to figure out $f(x, 1)$ and $f_Y(1)$. We know

$$f(x, 1) = f_X(x)f_Y(y = 1|X = x).$$

Continuous Bayes Theorem, contd'

$$f_Y(y) = \int_{-\infty}^{\infty} f(r,y)dr$$

$$\begin{aligned} f_Y(1) &= \int_{-\infty}^{\infty} f(r,1)dr \\ &= \int_{-\infty}^{\infty} f_X(r)f(y=1|X=r)dr. \end{aligned}$$

Therefore,

$$f_X(x|Y=1) = \frac{f_X(x)f_Y(y=1|X=x)}{\int_{-\infty}^{\infty} f_X(r)f_Y(y=1|X=r)dr}.$$

In general:

$$\begin{aligned} f_X(x|Y=y) &= \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(r,y)dr} \\ &= \frac{f_X(x)f_Y(y|X=x)}{\int_{-\infty}^{\infty} f_X(r)f_Y(y|X=r)dr}. \end{aligned}$$

The last equality is called **continuous Bayes theorem**.

Ex.4

Suppose that $f_X(x) = e^{-x} I(x > 0)$ and the conditional pdf of Y given $X = x$ is $f_Y(y|X = x) = xe^{-yx} I(y > 0)$. Find (a) $f_X(x|Y = y)$ and (b) $P(X > 1|Y = 1)$.

Solution: First we write down

$$f_X(x|Y = y) = \frac{f_X(x) f_Y(y|X = x)}{\int_{-\infty}^{\infty} f_X(r) f_Y(y|X = r) dr}$$

$$f_X(x) = e^{-x} I(x > 0).$$

$$f_Y(y|X = x) = ye^{-xy}.$$

$$f_X(x) f_Y(y|X = x) = xe^{-(1+y)x} I(x > 0).$$

Then we know

$$f_X(x|Y = y) = \frac{xe^{-(1+y)x} I(x > 0)}{\int_{-\infty}^{\infty} re^{-(1+y)r} I(r > 0) dr}.$$

$$
\begin{aligned}
& \int_{-\infty}^{\infty} re^{-(1+y)r} I(x>0)dr \\
= & \int_0^{\infty} re^{-(1+y)r} dr \\
= & \frac{1}{(1+y)^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_X(x|Y=y) &= \frac{xe^{-(1+y)x} I(x>0)}{\frac{1}{(1+y)^2}} \\
&= (1+y)^2 xe^{-(1+y)x} I(x>0).
\end{aligned}
$$

(b) By (a) we know

$$
f_X(x|Y=1) = 4xe^{-2x} I(x>0).
$$

$$
\begin{aligned}
P(X>1|Y=1) &= \int_1^{\infty} f_X(x|Y=1)dx \\
&= \int_1^{\infty} 4xe^{-2x} dx \\
&= 3e^{-2}
\end{aligned}
$$

Bivariate Transformation

Suppose we have a pair of random variables (X, Y). We create a new pair of random variables (U, V) by a bivariate transformation from (X, Y), that is,

$$U = g_1(X, Y) \quad V = g_2(X, Y).$$

Let's assume the transformation is one-to-one and smooth, which means there are h_1, h_2 such that

$$X = h_1(U, V) \quad Y = h_2(U, V)$$

and g_1, g_2, h_1, h_2 are differentiable.

Examples of Bivariate Transformation

Example 1:

$$U = X + Y \quad V = X - Y$$

then

$$X = \frac{U + V}{2} \quad Y = \frac{U - V}{2}$$

Example 2:

$$R = \sqrt{X^2 + Y^2} \quad \Theta = \text{arctan}\left(\frac{Y}{X}\right)$$

then

$$X = R\cos(\Theta) \quad Y = R\sin(\Theta)$$

Jacobian of Bivariate Transformation

Suppose we have the following one-to-one bivariate transformation from (X, Y) to (U, V)

$$U = g_1(X, Y) \quad V = g_2(X, Y).$$

Assume we can write

$$X = h_1(U, V) \quad Y = h_2(U, V).$$

Let

$$\mathbf{J} = \begin{bmatrix} \frac{\partial h_1(u,v)}{\partial u} & \frac{\partial h_1(u,v)}{\partial v} \\ \frac{\partial h_2(u,v)}{\partial u} & \frac{\partial h_2(u,v)}{\partial v} \end{bmatrix}$$

The Jacobian of the transformation is denoted by $|detJ|$, which is computed by

$$|\frac{\partial h_1(u,v)}{\partial u} \times \frac{\partial h_2(u,v)}{\partial v} - \frac{\partial h_1(u,v)}{\partial v} \times \frac{\partial h_2(u,v)}{\partial u}|.$$

Examples of Jacobian Calculation

Example 1:

$$U = X + Y \quad V = X - Y$$

then

$$X = \frac{U+V}{2} = h_1(U,V) \quad Y = \frac{U-V}{2} = h_2(U,V)$$

$$\frac{\partial h_1(u,v)}{\partial u} = \frac{1}{2} \quad \frac{\partial h_1(u,v)}{\partial v} = \frac{1}{2}$$

$$\frac{\partial h_2(u,v)}{\partial u} = \frac{1}{2} \quad \frac{\partial h_2(u,v)}{\partial v} = -\frac{1}{2}$$

$$|detJ| = |\frac{1}{2}\frac{1}{2} - (-\frac{1}{2})\frac{1}{2}| = \frac{1}{2}$$

Examples of Jacobian Calculation, contd'

Example 2:

$$R = \sqrt{X^2 + Y^2} \quad \Theta = \text{arctan}\left(\frac{Y}{X}\right)$$

then

$$X = R\cos(\Theta) = h_1(R, \Theta)$$

$$Y = R\sin(\Theta) = h_2(R, \Theta)$$

Think R is U and Θ is V.

$$\frac{\partial h_1(r,\theta)}{\partial r} = \cos(\theta) \quad \frac{\partial h_1(r,\theta)}{\partial \theta} = -r\sin(\theta)$$

$$\frac{\partial h_2(r,\theta)}{\partial r} = \sin(\theta) \quad \frac{\partial h_2(r,\theta)}{\partial \theta} = r\cos(\theta)$$

$$|detJ| = |\cos(\theta) r\cos(\theta) - (-r\sin(\theta))\sin(\theta)| = r$$

Joint pdf after bivariate transformation

Suppose we have the following one-to-one bivariate transformation from (X, Y) to (U, V)

$$U = g_1(X, Y) \quad V = g_2(X, Y).$$

Assume we can write

$$X = h_1(U, V) \quad Y = h_2(U, V).$$

Let $f_{X,Y}$ denote the joint pdf of (X, Y)

Let $f_{U,V}$ denote the joint pdf of (U, V)

Then the joint pdf of (U, V) can be computed by

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|detJ|$$

Example 1 and Example 2

Suppose the joint pdf of (X, Y) is $f(x, y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$. Let $(U, V) = (X+Y, X-Y)$ and $(R, \Theta) = (\sqrt{X^2+Y^2}, \text{arctan}\left(\frac{Y}{X}\right))$ Find the joint pdf of (U, V) and (R, Θ).

Solution:

$$f_{U,V}(u, v) = \frac{1}{2\pi}e^{-\frac{(\frac{u+v}{2})^2+(\frac{u-v}{2})^2}{2}}\frac{1}{2}$$

$$f_{U,V}(u, v) = \frac{1}{4\pi}e^{-\frac{u^2+v^2}{4}}$$

$$f_{R,\Theta}(r, \theta) = \frac{1}{2\pi}e^{-\frac{(r\cos(\theta))^2+(r\sin(\theta))^2}{2}}r$$

$$f_{R,\Theta}(r, \theta) = \frac{1}{2\pi}re^{-\frac{r^2}{2}}$$

$$r \geq 0, -\pi \leq \theta \leq \pi$$

So R and Θ are independent. Why? because of the factorization theorem.

What are the marginal pdfs of Θ and R?

$$f_\Theta(\theta) = \int_0^\infty \frac{1}{2\pi} r e^{-\frac{r^2}{2}} dr = \frac{1}{2\pi} e^{-\frac{r^2}{2}} |_0^\infty = \frac{1}{2\pi}$$

$$f_R(r) = \int_{-\pi}^{\pi} \frac{1}{2\pi} r e^{-r^2} d\theta = r e^{-\frac{r^2}{2}}$$

What is the marginal pdf of $T = \frac{R^2}{4}$? $R = \sqrt{4T}$

$$f_T(t) = \sqrt{4t} e^{-\frac{\sqrt{4t}^2}{2}} |\frac{d\sqrt{4t}}{dt}| = 2e^{-2t}$$

Joint distribution of n random variables

Notation: n random variables X_i $i = 1, 2, \ldots, n$
$(x_1, x_2, \ldots, x_n)$ is the value of the n-dimensional random vector $(X_1, X_2, \ldots, X_n)$.

Definition: A joint pdf function $f(x_1, \ldots, x_n)$ must satisfy two conditions

1. $f(x_1, \ldots, x_n) \geq 0$

2. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_1 \cdots dx_n = 1$.

Let A be some region, and denote $P(A) = P((X_1, \ldots, X_n) \in A)$. Then

$$P(A) = \int \cdots \int_{(x_1, \ldots, x_n) \in A} f(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

The joint distribution function is

$$F(x_1, \dots, x_n) = \mathrm{Pr}(X_1 \le x_1, \dots, X_n \le x_n).$$

$$F(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(r_1, \dots, r_n) dr_1 \cdots dr_n$$

$$f(x_1, \dots, x_n) = \frac{\partial^n F(x_1, \dots, x_n)}{\partial x_1 \cdots \partial x_n}$$

Marginal pdf

The marginal pdf of X_1 can be computed by

$$f_{X_1}(x_1) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1\ dimension} f(x_1, \dots, x_n) dx_2 \cdots dx_n.$$

Similarly,

$$f_{X_n}(x_n) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1\ dimension} f(x_1, \dots, x_n) dx_1 dx_2 \cdots dx_{n-1}.$$

Basically, to find the marginal pdf of X_i, we just take the integral of $f(x_1, \dots, x_n)$ with respect to all xs except x_i.

The conditional pdf of X_1 given that $X_2 = x_2, \ldots, X_n = x_n$ is

$$f_{X_1}(x_1 | X_2 = x_2, \ldots, X_n = x_n)$$

$$= \frac{f(x_1, x_2, \ldots, x_n)}{\int_{-\infty}^{\infty} f(r_1, x_2, \ldots, x_n) dr_1}$$

and

$$\int_{-\infty}^{\infty} f(r_1, x_2, \ldots, x_n) dr_1$$

gives us the joint pdf of $(X_2, X_3, \ldots, X_n)$.

The conditional joint pdf of (X_1, X_2) given that $X_3 = x_3, \ldots, X_n = x_n$ is

$$f_{(X_1, X_2)}(x_1, x_2 | X_3 = x_3, \ldots, X_n = x_n)$$

$$= \frac{f(x_1, x_2, \ldots, x_n)}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r_1, r_2, \ldots, x_n) dr_1 dr_2}$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r_1, r_2, \ldots, x_n) dr_1 dr_2$$

gives us the joint pdf of $(X_3, \ldots, X_n)$.

Independence

Definition: We say X_1, X_2, ..., X_n are independent if only if

$$f(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

That is, the joint pdf is equal to the product of the marginal densities.

If X_1, X_2, ..., X_n are independent, then

$$\begin{aligned} &P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n) \\ = &P(X_1 \in A_1)P(X_2 \in A_2) \cdots P(X_n \in A_n) \end{aligned}$$

IID RVs: If X_1, X_2, $\ldots X_n$ are independent and have the same distribution, that is, they have the same pdf. Then we say X_1, X_2, $\ldots X_n$ are independent and identically distributed (i.i.d.) random variables.

Factorization Theorem: X_1, X_2, ..., X_n are independent if only if

$$f(x_1, \ldots, x_n) = h_1(x_1) \cdots h_n(x_n).$$

Ex 1: Suppose the joint pdf of (X, Y, Z) is

$$f(x,y,z) = \begin{cases} e^{-x-y-z} & \text{if } 0 < x, y, z \\ 0 & \text{Otherwise} \end{cases}$$

Use the factorization theorem to show they are independent.

Solution:

$$\begin{aligned} f(x,y,z) &= e^{-x-y-z} I(x > 0, y > 0, z > 0) \\ &= e^{-x} e^{-y} e^{-z} I(x > 0) I(y > 0) I(z > 0). \end{aligned}$$

Let $h_1(x) = e^{-x} I(x > 0)$ $h_2(y) = e^{-y} I(y > 0)$ and $h_3(z) = e^{-z} I(z > 0)$. Then

$$f(x,y,z) = h_1(x) h_2(y) h_3(z)$$

By the factorization theorem we show X, Y, Z are independent.

Ex 2: Suppose X, Y, Z are i.i.d. random variables following standard normal distribution. Write down the joint pdf of (X, Y, Z).

Solution: The pdf of standard normal distribution is

$$f(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}$$

Therefore,

$$\begin{aligned} f(x, y, z) &= f_X(x) f_Y(y) f_Z(z) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\ &= (\frac{1}{\sqrt{2\pi}})^3 e^{-\frac{x^2+y^2+z^2}{2}} \end{aligned}$$

Ex 3: Suppose $X_1, \ldots, X_n$ are i.i.d. random variables following $Unif(0, a)$ distribution. Let $X_{(n)}$ be the maximum of $X_1, X_2, \ldots, X_n$. Find the pdf of $X_{(n)}$.

Solution: Note that since $0 < X_i < a$, we know $0 < X_{(n)} < a$. We compute $P(X_{(n)} \leq t)$ for any given t, $0 < t < a$

$$\begin{aligned} P(X_{(n)} \leq t) &= P(X_1 \leq t, X_2 \leq t, \ldots, X_n \leq t) \\ &= P(X_1 \leq t) P(X_2 \leq t) \cdots P(X_n \leq t) \end{aligned}$$

We used the independence assumption. Also by the identical distribution assumption, we know

$$P(X_i \leq t) = \int_0^t f(x)dx = \int_0^t \frac{1}{a} dx = \frac{t}{a}$$

for each $i = 1, 2, \ldots, n$.

$$P(X_{(n)} \leq t) = \left(\frac{t}{a}\right)^n$$

Thus

$$f_{X_{(n)}}(t) = \frac{dP(X_{(n)} \leq t)}{dt} = \frac{n}{a} \left(\frac{t}{a}\right)^{n-1}$$