Math 543/643: Stochastic Process

HW#1

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Problem 1. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y}e^{-y}}{y}, \quad x > 0, \quad y > 0$$

Find $E(X|Y)$.

Problem 2. Let X be exponential with mean $1/\lambda$, i.e.

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

Calculate $E(X|X > 1)$.

Problem 3. Let Y be a gamma random variable with parameters (s, α), that is

$$f(y) = Ce^{-\alpha y}y^{s-1}, \quad y > 0,$$

where $C > 0$ is a constant which does not depend on y. Suppose also that the conditional distribution of X given $Y = y$ is Poisson with mean y, i.e.

$$P(X = i|Y = y) = e^{-y}y^i/i!, \quad i \geq 0.$$

Show that the conditional distribution of Y given $X = i$ is the gamma distribution with parameters $(s + i, \alpha + 1)$.

Problem 4. (a) Prove

$$Cov(X, Y) = Cov(X, E[Y|X])$$

(b) Suppose for constants a and b, we have

$$E[Y|X] = a + bX$$

Prove

$$b = \frac{Cov(X, Y)}{Var(X)},$$

what connection can you make between this and linear regression? [If you did not take MTH547, please Google simple linear regression.]

Problem 5. (a). Prove $E[Y^2|X] \geq (E[Y|X])^2$.

(b). If $E[Y|X] = 1$, prove $Var(XY) \geq Var(X)$.

Problem 6. Let $S_0 = 0$, and for $n \geq 1$, let $S_n = X_1 + \cdots + X_n$ be the sum of n i.i.d. exponential R.V. with mean 1. Show that

$$Y_n = 2^n \exp(-S_n), \quad n \geq 0$$

defines a nonnegative martingale.

Problem 7. Consider a stochastic process (take integer values) that evolves according to the following law: if $X_n = 0$, then $X_{n+1} = 0$; if $X_n > 0$, then

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability} \quad 1/2 \\ X_n - 1 & \text{with probability} \quad 1/2 \end{cases}$$

X_n can be thought of the fortune of a player of a fair game who wagers $1 at each bet and who is forced to quit if all money is gone ($X_n = 0$).

(a) Show that X_n is a nonnegative martingale.

(b) Suppose that $X_0 = i > 0$, use the maximal inequality to bound

$$P(X_n \geq N \quad \text{for some} \quad n \geq 0 | X_0 = i).$$