Math 543/653: Stochastic Process

HW#3

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Problem 1. Let $X(t)$ be a Poisson process of rate $\lambda > 0$. Validate the identity

$$\{W_1 > w_1, W_2 > w_2\}$$

if and only if

$$\{X(w_1) = 0, X(w_2) - X(w_1) = 0 \text{ } or \text{ } 1\}.$$

Use this to determine the joint upper tail probability

$$\begin{aligned} P(W_1 > w_1, W_2 > w_2) &= P(X(w_1) = 0, X(w_2) - X(w_1) = 0 \text{ } or \text{ } 1) \\ &= e^{-\lambda w_1}[1 + \lambda(w_2 - w_1)]e^{-\lambda(w_2 - w_1)}. \end{aligned}$$

Finally, differentiate twice to obtain the joint density function

$$f(w_1, w_2) = \lambda^2 e^{-\lambda w_2} \quad for \quad 0 < w_1 < w_2.$$

Verify it is indeed a p.d.f.

Problem 2. The joint probability density function for the waiting times W_1 and W_2 is given by

$$f(w_1, w_2) = \lambda^2 e^{-\lambda w_2} \quad for \quad 0 < w_1 < w_2.$$

Please give the conditional probability density function for W_1 given that $W_2 = w_2$.

Problem 3. The joint probability density function for the waiting times W_1 and W_2 is given by

$$f(w_1, w_2) = \lambda^2 e^{-\lambda w_2} \quad for \quad 0 < w_1 < w_2.$$

Please determine the marginal distributions of W_1 and W_2.

Problem 4. Let $\{W_n\}$ be the sequence of waiting times in a Poisson process with rate $\lambda = 1$. Show that $X_n = 2^n \exp(-W_n)$ defines a nonnegative martingale.

Problem 5. Let $X(t)$ be a Poisson process of rate $\lambda > 0$. Determine the cumulative distribution function of the gamma density as a sum of Poisson probabilities by first verifying and they using the identity $W_r \leq t$ if and only if $X(t) \geq r$.

Problem 6. Let $N(t)$ be a Poisson process with rate $\lambda > 0$, which is independent of a nonnegative random variable T with mean $E(T) = \mu$ and $Var(T) = \sigma^2$. Find $Cov(T, N(T))$ and $Var(N(T))$.

Problem 7. Let $W_1, W_2, \cdots$ be the event times in a Poisson process $X(t)$ with rate $\lambda > 0$, and let $f(w)$ be an arbitrary function. Prove

$$E\left[\sum_{i=1}^{X(t)} f(W_i)\right] = \lambda \int_0^t f(w)dw.$$

Problem 8. Let $N(t)$ be a Poisson process with rate $\lambda > 0$. Suppose that we observed n events at time t, for $0 < u < t$, let X be the number of events observed during $(u, t]$. What is the distribution of X? Prove your answer.