Math 543/653: Stochastic Modeling)

HW #4

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Problem 1. A Markov Chain $X_0, X_1, X_2, \cdots$ has the transition probability matrix

$$P = \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\|$$

The Markov chain has state space $\{0, 1, 2\}$.

(a). Determine the conditional probability $P(X_3 = 1|X_0 = 0)$ and $P(X_3 = 1|X_1 = 0)$.

(b). The initial distribution is $p_0 = 0.5$ and $p_1 = 0.5$. Please find

$$P(X_0 = 1, X_1 = 1, X_2 = 0)$$

and

$$P(X_1 = 1, X_2 = 1, X_3 = 0)$$

(c). The initial distribution is $p_0 = 0.5$ and $p_1 = 0.5$. Please determine the probabilities $P(X_2 = 0)$ and $P(X_3 = 0)$.

Problem 2. Consider a Markov chain with state space $\{0, 1, 2, 3\}$. Suppose the transition probability matrix is

$$P = \left\| \begin{array}{cccc} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{array} \right\|.$$

If the initial distribution is $p_i = 1/4$, for $i = 0, 1, 2, 3$. Show that for all n, $P(X_n = k) = 1/4$, for $k = 0, 1, 2, 3$. Can you generalize this result from this example? Prove your conjecture.

Problem 3. Consider a Markov chain with state space $\{0, 1, 2, 3\}$, and the transition probability matrix is

$$P = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\|.$$

(a) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(b) Determine the mean time to absorption.

Problem 4. Consider a Markov chain with state space $\{0, 1, 2, 3, 4\}$, and the transition probability matrix is

$$P = \begin{Vmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

where $p + q = 1$. Determine the mean time to reach state 4 starting from state 0. That is, find $E[T|X_0 = 0]$, where $T = \min\{n \geq 0; X_n = 4\}$.

Problem 5. Consider a Markov chain with state space $\{0, 1, 2\}$, and the transition probability matrix is

$$P = \begin{Vmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{Vmatrix}.$$

The process starts with $X_0 = 0$. Eventually the process will end up in state 2. What is the probability that when the process moves into state 2, it does so from state 1?

Hint: *let* $T = \min\{n \geq 0; X_n = 2\}$, *then the asked case happens if* $X_{T-1} = 1$, *considering the initial states, we want to find* $P(X_{T-1} = 1|X_0 = 0)$. *Let* $z_i = P(X_{T-1} = 1|X_0 = i)$, *use first step analysis to find the desired result.*

Problem 6. Consider a general one dimensional random walk on $\{0, 1, 2, \cdots, N\}$ with transition probability matrix

$$P = \begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_2 & r_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{Vmatrix}$$

where $p_k > 0$, $q_k > 0$, and $r_k \geq 0$ for $k = 1, 2, \cdots, N-1$. Let $T = \min\{n \geq 0; X_n = 0 \quad or \quad X_n = N\}$. Consider the probability of gambler's ruin

$$u_i = P(X_T = 0|X_0 = i).$$

Use first step analysis to prove that, for $i = 1, \cdots, N-1$:

$$u_i = \frac{\rho_i + \cdots + \rho_{N-1}}{1 + \rho_1 + \rho_2 + \cdots + \rho_{N-1}}$$

where

$$\rho_k = \frac{q_1 q_2 \cdots q_k}{p_1 p_2 \cdots p_k}.$$

Problem 7. The number of offspring of an individual in a population is 0, 1, or 2 with probabilities $a > 0$, $b > 0$, and $c > 0$, $a + b + c = 1$. Express the mean and variance of the population in step n.

Problem 8. Let $\{X_n\}$ be a branching process with $E(\xi) = \mu$. Show that $Z_n = X_n/\mu^n$ is a nonnegative martingale.