Math 543/653: Stochastic Process

HW #6

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Problem 1. A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1, \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 5$. Let W_3 be the random time that it takes the process to reach state 3.

(a) Write W_3 as a sum of sojourn times and thereby deduce the mean time is $E(W_3) = 11/6$.

(b) Determine the mean of $W_1 + W_2 + W_3$.

(c) what is the variance of W_3?

Problem 2. Consider a pure birth process on the states $\{0, 1, \cdots, N\}$ for which $\lambda_k = (N-k)\lambda$ for $k = 0, 1, \cdots, N$. Suppose $X(0) = 0$, please determine the functions $P_n(t) = P(X(t) = n$ for $n = 0, 1$ and 2.

Problem 3. Consider a pure birth process on the states $\{0, 1, 2, \cdots\}$ for which $\lambda_k = \lambda$ for $k = 0, 1, 2, \cdots$, please show that

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, \cdots$$

Compare this to Poisson process.

Start of Background paragraph

In Class, we discussed the Markov process $\{X(t)\}$ with state $\{0, 1\}$ and infinitesimal matrix

$$A = \left\| \begin{array}{cc} -\alpha & \alpha \\ \beta & -\beta \end{array} \right\|,$$

and we derived the transition probability functions are

$$P_{00}(t) = (1-\pi) + \pi e^{-\tau t}, \qquad P_{01}(t) = \pi - \pi e^{-\tau t};$$

$$P_{10}(t) = (1-\pi) - (1-\pi)e^{-\tau t}, \qquad P_{11}(t) = \pi + (1-\pi)e^{-\tau t};$$

where $\pi = \alpha/(\alpha+\beta)$ and $\tau = \alpha + \beta$.

End of Background paragraph

Problem 4. Let $\{V(t)\}$ be the two state Markov process described in the Background paragraph. Suppose that that initial distribution is $(1-\pi, \pi)$, that is, $P(V(0) = 0) = 1 - \pi$ and $P(V(0) = 1) = \pi$. In this case, prove that $P(V(t) = 1) = \pi$ for all time $t > 0$.

Problem 5. Let $\{V(t)\}$ be the two state Markov process described in the Background paragraph. Suppose that that initial distribution is $(1-\pi, \pi)$, that is, $P(V(0) = 0) = 1 - \pi$ and $P(V(0) = 1) = \pi$. Prove that for $0 < s < t$,

$$E[V(s)V(t)] = \pi - \pi P_{10}(t-s)$$

and

$$Cov[V(s), V(t)] = \pi(1-\pi)e^{-\tau(t-s)}$$

Problem 6. The velocity $V(t)$ of a stop-and-go traveler is described by the two state Markov process given in the Background paragraph. The distance traveled in time t is the integral of the velocity

$$S(t) = \int_0^t V(u)du.$$

Assuming that the velocity at time $t = 0$ is $V(0) = 0$, determine the mean of $S(t)$. Take for granted that we can exchange the order of integral and expectation.