Math 543/653: Stochastic Process

HW #7

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Problem 1. Find the covariance functions for the following stochastic processes, where $B(t)$ is a standard Brownian motion.

(a) $U(t) = e^{-t}B(e^{2t})$ for $t \geq 0$.

(b) $V(t) = (1-t)B(t/(1-t))$ for $0 < t < 1$.

(c) $W(t) = tB(1/t)$, with $W(0) = 0$.

Problem 2. Consider a standard Brownian motion $\{B(t), t \geq 0\}$ at times $0 < u < u+v < u+v+w$ where $u, v, w > 0$.

(a) What is the probability distribution of $B(u) + B(u+v)$?

(b) What is the probability distribution of $B(u) + B(u+v) + B(u+v+w)$?

Problem 3. Calculate $E[e^{\lambda B(t)}]$ for an arbitrary constant λ and standard Brownian motion $B(t)$.

Problem 4. Consider standard Brownian motion $B(t)$ with $B(0) = 0$, let $M_t = \max\{B(u); 0 \leq u \leq t\}$.

(a) Evaluate $P(M_4 \leq 2)$.

(b) Find the number c such that $P(M_9 > c) = 0.1$.

Problem 5. (a) Let τ_0 be the largest zero of a standard Brownian motion not exceeding $a > 0$, i.e. $\tau_0 = \max\{u \geq 0; B(u) = 0 \textit{ and } u \leq a\}$, prove that

$$P(\tau_0 < t) = \frac{2}{\pi} \arcsin \sqrt{t/a}$$

a) Let τ_1 be the smallest zero of a standard Brownian motion exceeding $b > 0$, i.e. $\tau_0 = \min\{u \geq 0; B(u) = 0 \textit{ and } u \geq b\}$, prove that

$$P(\tau_1 < t) = \frac{2}{\pi} \arccos \sqrt{b/t}$$

Problem 6. (a) Find the conditional probability that a standard Brownian motion is not zero in the interval $(t, t+b]$ given that it is not zero in the interval $(t, t+a]$, where $0 < a < b$ and $t > 0$.

(a) Find the conditional probability that a standard Brownian motion is not zero in the interval $(0, b]$ given that it is not zero in the interval $(0, a]$, where $0 < a < b$. *[Hint: Let* $t \to 0$ *in the result of (a).]*