“Challenges are beautiful opportunities in disguise.” ...

- Normal Distribution.
- How to Solve General Normal Curve Problems?
Learned so far…

- Basic Probability Rules
- Discrete Random Variables
- Binomial Distribution
- Continuous Random Variables
- Uniform Distribution
Normal Random Variable

- A special case of *bell-shaped* distributions is a **normal distribution**.
Why normal distribution?

- Normal distribution is symmetric, bell-shaped, centered at the mean \( \mu \) and its spread is determined by the standard deviation \( \sigma \).

- Many psychological and educational variables are approximately normal.
- Many statistical methods and tests are derived for normal distribution.
- Empirical Rule is derived for Normal Distribution.
Empirical Rule

- About 68% of the data falls within one standard deviation (σ) from the mean (μ).
- About 95% of the data falls within two standard deviations from the mean.
- About 99.7% of the data falls within three standard deviations from the mean.

The empirical rule, also known as the 68-95-99.7 rule, describes the spread of data in a normal distribution.
Question: How to compute the probability that a normal random variable $X$ with mean $\mu$ and standard deviation $\sigma$ takes values in the interval $(a, b]$?

$$P(a < X < b) = ?$$

Answer: By integration or by using the tables of the standard normal distribution.
### Standard Normal Distribution, $N(0,1)$

<table>
<thead>
<tr>
<th>$Z$</th>
<th>Area below $Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.50</td>
<td>0.0062</td>
</tr>
<tr>
<td>-2.49</td>
<td>0.0064</td>
</tr>
<tr>
<td>-2.48</td>
<td>0.0066</td>
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<td>-2.47</td>
<td>0.0068</td>
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<tr>
<td>-2.46</td>
<td>0.0069</td>
</tr>
<tr>
<td>-2.45</td>
<td>0.0071</td>
</tr>
</tbody>
</table>

Shaded area: 0.006210
Normal Distribution Probabilities

Compute probabilities, \( P(X<a), P(x>b), P(a<X<b) \)

- **KEY IDEA:** any value from \( X \sim N(\mu, \sigma) \) can be transformed into the value on a standard normal distribution, called **z-score**:

\[
z = \frac{\text{value} - \text{mean}}{\text{Standard deviation}} = \frac{x - \mu}{\sigma}
\]

- **z** has a standard normal distribution, \( N(0, 1) \)
Normal Distribution Probabilities

- X has a normal distribution with a mean of 50 and a standard deviation of 10.
- What portion of values of this distribution is below 55.4?

- KEY IDEA: convert $X \sim N(50,10)$ to $Z \sim N(0,1)$

$$P(X \leq 55.4) = P\left(Z \leq \frac{55.4 - \mu}{\sigma}\right) = P\left(Z \leq \frac{55.4 - 50}{10}\right) = P(Z \leq 0.54)$$
### Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE $z$-values are shown in the following table:

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Normal Distribution Probabilities

- What portion of values of this distribution is below 55.4 and above 51.2?

\[ P(a < X \leq b) = P(X \leq b) - P(X \leq a) \]

\[ P(X \leq a) = P \left( Z \leq \frac{a - \mu}{\sigma} \right) = P(Z \leq z) \]
# Standard Normal Cumulative Probability Table

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Normal Distribution Probabilities

- What portion of values of this distribution is below 55.4 and above 51.2?

- $b = 55.4$

\[
P(X \leq 55.4) = P\left(Z \leq \frac{55.4 - 50}{10}\right) = P(Z \leq 0.54) = 0.7054
\]

- $a = 51.2$

\[
P(X \leq 51.2) = P\left(Z \leq \frac{51.2 - 50}{10}\right) = P(Z \leq 0.12) = 0.5478
\]

\[
P(51.2 \leq X \leq 55.4) = 0.7054 - 0.5478 = 0.1576
\]
Normal Distribution Probabilities

What portion of values of this distribution is above 51.2?

\[ P(X \geq b) = 1 - P(X \leq b) \]
Normal Distribution Probabilities

- What portion of values of this distribution is above 51.2?

\[ P(X \geq b) = 1 - P(X \leq b) \]

\[ P(X \geq 51.2) = 1 - 0.5478 = 0.4522 \]
Normal Distribution Percentiles

- What is the 75\textsuperscript{th} percentile of the N(50,10) distribution?
- KEY IDEA: find the 75\textsuperscript{th} percentile for N(0,1) and use:

$$X = \mu + z\sigma$$
Normal Distribution Percentiles

To find the 75\textsuperscript{th} percentile for N(0,1):

A. Use the standard normal distribution table
B. Look for the closest value to 0.75 (among the probability values)
C. Identify the corresponding z-score.
# Standard Normal Table

## Standard Normal Cumulative Probability Table

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For 0.7517 the corresponding z-score is 0.68

\[ P(Z \leq 0.68) = 0.7517 \]

The 75\textsuperscript{th} percentile, or Q3, is:
Normal Distribution Percentiles

- For 0.7517 the corresponding z-score is 0.68
  
  \[ P(Z \leq 0.68) = 0.7517 \]

- The 75\textsuperscript{th} percentile, or Q3, is:

  \[ X = \mu + z\sigma = 50 + 0.68 \times 10 = 56.8 \]
Example

Suppose that lengths of tails of adult Ring-tailed Lemurs are normally distributed with mean 50 cm and standard deviation 5 cm.
Example

- Suppose that lengths of tails of adult Ring-tailed Lemurs are $N(50 \text{ cm}, 5 \text{ cm})$.

- (a) What is the probability that a randomly selected adult ring-tailed lemur has a tail that is 45 inches $\text{ cm}$ or shorter?
Example

- Suppose that lengths of tails of adult Ring-tailed Lemurs are $N(50 \text{ cm}, 5 \text{ cm})$.

- (b) What is the probability that a randomly selected adult lemur has a tail that is 55 cm or longer?
Example

- Suppose that lengths of tails of adult Ring-tailed Lemurs are $N(50 \text{ cm}, 5 \text{ cm})$.
- (c) Complete the sentence. Only 10% of all adult ring-tailed lemur population have a tail that is ________cm or longer?
Example

- Suppose that lengths of tails of adult Ring-tailed Lemurs are $N(50 \text{ cm}, 5 \text{ cm})$.

- (d) Two adult ring-tailed lemurs will be randomly selected. What is the probability that both lemurs will have a tail that is 55 cm or longer?
QQ Plot for Checking Normality

- Q-Q Plot is a plot of the percentiles of a standard normal distribution against the corresponding percentiles of the observed data.

- If the observations follow approximately a normal distribution, the resulting plot should be roughly a straight line with a positive slope.
QQ Plot for Checking Normality

Normal Q-Q Plot of Hours Spent Sleeping
Thank you!