A multi-scale approach to the Erdős-Falconer type single distance problems.

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Background

- Problems
- A definition

Multi-scale approach

- A convoluted definition
- Results

Discrete single distance problem

- Conjecture[Erdős, 1946]: If $E \subset \mathbb{R}^2$ is a set of n points, then no distance may occur more than $\gtrsim n\sqrt{\log n}$ times.

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- Theorem,[Spencer, Szemerédi, Trotter, 1984]: For $d = 2$, no distance may occur more than $\gtrsim n^{\frac{4}{3}}$ times.

Discrete single distance problem

- Lenz (Private Communication to Erdős): Discrete, Euclidean, $d \geq 4$.
  Arrange half of the points on a unit circle in two dimensions, and the other half on a unit circle in two other dimensions. A single distance will occur n^2 times.

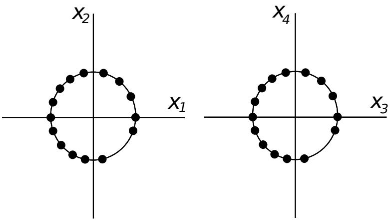

Figure: Lenz's example

Falconer distance problem

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  - Theorem,[Wolff 1999]: For $d = 2$, $s > \frac{4}{3}$ suffices.
  - Theorem,[Erdoğan 2006]: For $d \geq 3$, $s > \frac{d}{2} + \frac{1}{3}$, suffices.

Continuous single distance problem

- Main estimate: If μ is a Borel probability measure, supported on $E \subset [0,1]^d$, with Hausdorff dimension $s > \frac{d+1}{2}$, and satisfying an energy estimate, $I_s = \int \int |x-y|^{-s} d\mu(x) d\mu(y) < \infty$, then

$$(\mu \times \mu)\{(x,y) \in E \times E : 1 \leq |x-y| \leq 1+\epsilon\} \lesssim \epsilon.$$

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- False! Mattila 1985: Continuous, Euclidean, $d = 2$. Consider $[0,1] \times \mathcal{C}_\alpha$, where $0 < \alpha < \frac{1}{2}$. We have bounded energy, but we don't get the "Main estimate."
  Can be adapted for $d = 3$, but not higher dimensions.

s-adaptability

- Definition

  We say that a set P of n points in $[0,1]^d$ is *s-adaptable*, for $0 \leq s \leq d$, if it satisfies the following two conditions:

$$i)\frac{1}{\binom{n}{2}} \sum_{x \neq y \in P} \frac{1}{|x-y|^s} \lesssim 1.$$

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  - Any subset is 0-adaptable
  - Any d-adaptable subset is basically a perturbed lattice.

Conversion mechanism

- Take a set in $E \subset [0,1]^d$ of n, which is "s-adaptable" (basically a discrete energy condition), and thicken each point by $\epsilon = n^{-\frac{1}{s}}$. The same continuous analysis techniques now apply directly to the thickened set.

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- Theorems,[Iosevich, Jorati, Łaba, 2007]: For $d \geq 2, s > \frac{d+1}{2}$, $E \subset \mathbb{R}^d$ of n points, "homogeneous", with "s-well-behaved" metrics $\Rightarrow$ no single distance occurs more than $n^{2-\frac{1}{s}}$ times.

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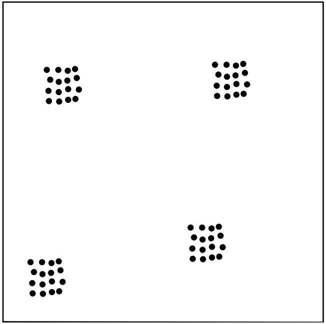
- 

Figure: Concentrated clusters of points

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- (iii)The points of E within each Q_j are distributed to form a v-adaptable set *within* the Q_j.

An example of a $(u, v; \alpha)$-adaptable set

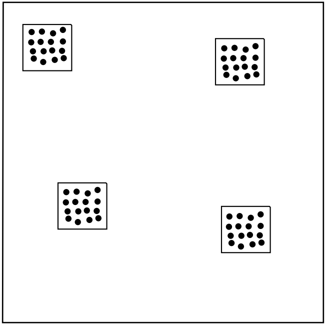

Figure: Cover of disjoint cubes (i)

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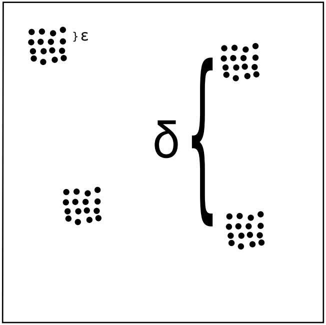

Figure: The centers of the clusters are u-adaptable on scale δ (ii), and each cluster is v-adaptable on scale ϵ (iii).

Main results

- Theorem
  Let E be a $(u, v; \alpha)$-adaptable set of N points, where $u \geq \frac{d-1}{2}$, and $t = \min\{1, u - \frac{d-1}{2}\}$ then for $\epsilon = N^{-\frac{\alpha u + (1-\alpha)v}{uv}}$, we have
  $|\{(x, y) \in E \times E : 1 \leq |x - y| \leq 1 + \epsilon\}| \lesssim N^{2-\left(2\alpha + \frac{t-\alpha}{u} - \frac{d-1}{v}\alpha\right)}$

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  Many new non-trivial estimates for the range $s < \frac{uv}{\alpha u + (1-\alpha)v}$.

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- Corollary (one illustrative example)
  Many new non-trivial estimates for the range $s < \frac{uv}{\alpha u + (1-\alpha)v}$.
  - In particular, we now have non-trivial estimates for a wide class of sets which are only s-adaptable for $s < \frac{d-1}{2}$.

A word on the proof

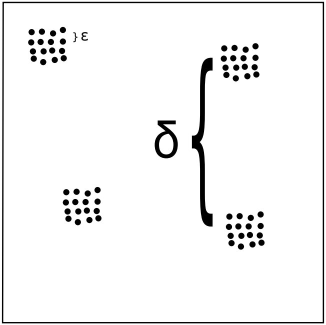

Figure: Simple combinatorics on scale ϵ, Falconer's estimate on scale δ.

Thanks

Thank You!