5 Maximal functions

5.1 Covering lemmas

Lemma 5.1 (Vitali)**.** Let $\{B_j : j \in J\}$ be a finite collection of finite open balls in $\mathbb{R}^d$. There exists a subcollection $K \subset J$ such that the B_{j_k} are disjoint and

$$\bigcup_j B_j \subset \bigcup_k 3B_{j_k}. \tag{5.1}$$

Proof. Suppose that B_{j_1} be the largest ball with the smallest index. Let F_1 denote the set of balls that are NOT disjoint from B_{j_1}. Now, select B_{j_2} as the largest ball with the smallest index that has not already been put in F_1, and has $j_1 \neq j_2$. Define F_2 to be the set of unselected balls which are not disjoint from B_{j_2}. Proceed inductively by selecting B_{j_k} as the largest ball with the smallest index that has not already been put in any F_l, and has $j_k \neq j_l$ for all $l < k$, and defining F_k to be the set of unselected balls which are not disjoint from B_{j_k}. When there all of the balls are either selected or in at least one F_k, we stop. Now, it is clear that all of the B_{j_k} are in $\bigcup_k 3B_{j_k}$. So, any ball, B_0, that was not selected must be in some F_l. Suppose that l' is the minimum of the set of l for which $B_0 \in F_l$. Therefore, it touches $B_{j_{l'}}$ AND has a diameter less than or equal to it. So, by the triangle inequality, it must be contained in $3B_{j_{l'}}$. $\square$

Lemma 5.2 (Whitney)**.** For every open $E \subset \mathbb{R}^d$, there exists a partition of E into countably many dyadic cubes such that the diameter of each non-unit cube is bounded above and below by its distance to the boundary of E.

Proof. Divide E into d-dimensional unit cubes whose boundaries line up with the coordinate axes. Let K_0 denote the (possibly empty) set of these cubes whose distances from ∂E (the boundary of E) are at least $6\sqrt{d}$. Now, suppose that K_{s-1} has been constructed. To construct K_s, take every cube which is not yet in some K_j and divide it evenly into 2^d subcubes of side-length 2^{-s}. Now, let K_s be the subset of these new subcubes whose distances from ∂E are at least $2^{-s} \cdot 6\sqrt{d}$.

We can see that for any $s \geq 1$ and $C \in K_s$, the distance from C to ∂E is $\leq 2^{-s} \cdot 18\sqrt{d}$, because it is a subcube of a larger cube which was not in K_{s-1}, whose distance from ∂E was $\leq 2^{-s} \cdot 6\sqrt{d}$, and $18 = 6 \cdot 2 + 6 \cdot 2 \cdot \frac{1}{2}$.

As the sets K_s are subsets of E, it is clear that $\bigcup_s K_s \subset E$. To get the reverse inclusion, we recall that E is open, and therefore has no point of ∂E. Therefore, every point in E is some positive distance from from the boundary, and by construction, in a cube of diameter proportional to that distance, which is in some K_s.

□

5.2 Maximal functions

Often times we want to estimate something large, and directly employing L^1 estimates does not work. In many cases, if we average in a more careful fashion, we can get sharper estimates. Consider the following estimate.

Exercise 5.1 (pigeonholing, again)**.** Prove that for non-negative $f \in L^1(\mathbb{R})$, there must be an x in the support of f such that

$$f(x) \leq \frac{\|f\|_1}{|f^{-1}((0,\infty))|}.$$

Definition 5.3. Let $B(x,\delta)$ denote the open ball of radius δ centered at x. Given a function, f, we define the *δ-average of f at x* to be

$$A_\delta(f)(x) = \frac{1}{|B(x,\delta)|}\int_{B(x,\delta)} f(y)d\mu(y).$$

Similarly, when it is clear from context, we define the average over a set E

$$A_E(f)(x) = \frac{1}{|E|}\int_E f(y)d\mu(y).$$

Definition 5.4. For a function, f, the *Hardy-Littlewood maximal function (or operator)* of f at x is

$$M_c(f)(x) = \sup_{\delta>0}\left\{A_\delta(|f|)(x)\right\}.$$

The *uncentered maximal function* is the same, but with the δ-balls in the average not necessarily centered at x.

$$M(f)(x) = \sup_{\substack{\delta>0 \\ |x-y|<\delta}}\left\{A_\delta(|f|)(y)\right\}.$$

Exercise 5.2. Show that $M_c(f) \leq M(f)$.

Exercise 5.3. Show that for $f \in L^1(\mathbb{R})$, which is not constantly zero, $M(f) \notin L^1(\mathbb{R})$. Hint: Prove that if f is supported in a compact set E, then for any $x \notin E$,

$$M_c(f)(x) \gtrsim \frac{\|f\|_1}{(|x| + \operatorname{diam}(E))^d}.$$

Theorem 5.5. *M is weak type (1,1) (i.e. it maps $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$).*

Proof. For any $\alpha > 0$, define $E_\alpha = \{x \in \mathbb{R}^d : |M(f)(x)| > \alpha\}$. This set is open (prove it!). Let K be a compact subset of E_α. Now, notice that for each $x \in K$, there must be an open ball, $B_x \ni x$, with $A_{B_x}(|f|) > \alpha$. As K is compact, there must be a finite subcover of K, which we call $\{B_j\}$. Using Vitali's covering lemma, we find a subcollection, $\{B_{j_k}\}$, satisfying (5.1). Now, we get

$$|K| \leq \sum_j |B_j| \leq 3^d \sum_k |B_{j_k}| \leq \frac{3^d}{\alpha} \sum_{j_k} \int_{B_{j_k}} |f(y)|dy \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)|dy,$$

which tells us that

$$\alpha|K| \lesssim \|f\|_1.$$

As the Lebesgue measure is sufficiently well-behaved (it is a so-called *regular* measure), we can approximate, up to an arbitrary level of accuracy, the measure of any open set by the measure of some compact set contained in it. Now take the supremum over all compact sets $K \subset E_\alpha$ to get $\alpha|E_\alpha| \lesssim \|f\|_1$. So, for all $\alpha > 0$, we have

$$\|f\|_1 \gtrsim \alpha|E_\alpha| = \alpha|\{x \in \mathbb{R}^d : |M(f)(x)| > \alpha\}| = \alpha d_{M(f)}(\alpha).$$

Since this holds for any $\alpha > 0$, it holds for the supremum over all α and we get the weak L^1 estimate. □

Exercise 5.4. Prove that M maps $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.

Corollary 5.6. For $1 < p \leq \infty, M$ maps $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.

Proof. By Exercise 5.4, we have that M maps $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. By Theorem 5.5, we have a weak type (1,1) estimate. Notice that M is defined on $L^1_{loc}(\mathbb{R}^d) \supset L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$. By Marcinkiewicz interpolation, we get that M maps $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$. □