6 Fourier Transform

6.1 Fourier transform

Definition 6.1. If $f \in L^1(\mathbb{R}^d)$, then its *Fourier transform* is $\hat{f} : \mathbb{R}^d \to \mathbb{C}$, defined

$$\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx.$$

We can compute $\hat{\mu}$ as well. We get

$$\hat{\mu}(\xi) = \int e^{-2\pi i x \cdot \xi} d\mu(x).$$

Exercise 6.1. Let $\delta_a(E)$ be the *Dirac measure* at a, where $\delta_a(E) = 1$ if $a \in E$, and zero otherwise. Show $\hat{\delta}_a(\xi) = e^{-2\pi i a \cdot \xi}$.

Exercise 6.2. Show that for $x \in \mathbb{R}$, we have $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$.

Proposition 6.2. Let $\Gamma(x) = e^{-\pi |x|^2}$ be the *Gaussian*. Then $\hat{\Gamma}(\xi) = e^{-\pi |\xi|^2}$.

Proof. Suppose that we are in $\mathbb{R}$. If we were in higher dimensions, the integral would factor down to one-dimensional terms anyway.

$$\begin{aligned} \hat{\Gamma}(\xi) &= \int_{-\infty}^{\infty} e^{-2\pi i x \xi} e^{-\pi x^2} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} dx e^{-\pi \xi^2} \\ &= \int_{-\infty + i\xi}^{\infty + i\xi} e^{-\pi x^2} dx e^{-\pi \xi^2} = I \cdot e^{-\pi \xi^2}. \end{aligned} \tag{6.1}$$

Now we do some contour integration in the complex plane to evaluate I. Fix an $R > 0$. Let P_R denote the contour in the complex plane described by connecting the four points $(-R, 0), (R, 0), (R, \xi)$, and $(-R, \xi)$ clockwise. Since the Gaussian is entire, we know that the integral over P_R is zero. Calculating explicitly, we see that

$$\begin{aligned} 0 = \oint_{P_R} e^{-\pi z^2} dz = \int_{-R}^{R} e^{-\pi x^2} dx + \int_{-R+i\xi}^{R+i\xi} e^{-\pi x^2} dx \\ + \int_{\xi}^{0} e^{-\pi (R - iy)^2} dy + \int_{0}^{\xi} e^{-\pi (R + iy)^2} dy. \end{aligned}$$

Now, the last two terms cancel. Taking the limit as $R \to \infty$ will give us that $I = 1$. □

6.2 Schwarz class

Definition 6.3. Given a point $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$, we use the notation

$$x^\alpha = \prod_j x_j^{\alpha_j},$$

and call α a *multiindex*. The *length* of α is

$$|\alpha| = \sum_j \alpha_j.$$

Similarly, for a multiindex β, a partial differential operator will be written

$$D^\beta = \prod_j \frac{\partial}{\partial x_j^{\beta_j}}.$$

Definition 6.4. The *Schwarz space*, denoted $\mathcal{S}$, is the space of functions $f : \mathbb{R}^d \to \mathbb{C}$ such that f is C^∞ and $x^\alpha D^\beta f$ is bounded for all multiindices α, β. We call the following *Schwarz seminorms*:

$$\|f\|_{\alpha\beta} = \|x^\alpha D^\beta f\|_\infty.$$

Proposition 6.5. C_0^∞ is dense in $\mathcal{S}$.

Proposition 6.6. The following are equivalent:

- i) $f \in \mathcal{S}$
- ii) $(1 + |x|)^N D^\beta f$ is bounded for any multiindex β and $N \in \mathbb{N}$
- iii) $\lim_{x\to\infty} x^\alpha D^\beta f = 0$ for any multiindices α and β.

6.3 Convolution

Given two suitable functions, f and g, we define the *convolution* of f and g to be

$$(f * g)(x) = \int f(y)g(x - y)dy.$$

Exercise 6.3. Prove that convolution is commutative.

Lemma 6.7. If $\phi \in C_0^\infty$ and $f \in L^1_{loc}$, then $\phi * f \in C^\infty$ and

$$D^\alpha(\phi * f) = (D^\alpha \phi) * f.$$

Proof. At the moment, we need only prove this for multiindices α with $|\alpha| = 1$, because we can use induction to prove the result for general α. Since we are restricting ourselves to one derivative at a time, to simplify exposition, suppose that f and ϕ both map $\mathbb{R} \to \mathbb{R}$. In general, we can just analyze one coordinate at a time.

Now consider difference quotients of the form

$$d(h) = \frac{1}{h}\left((\phi * f)(x+h) - (\phi * f)(x)\right)$$

By the exercise above, we get

$$\begin{aligned} d(h) &= \frac{1}{h}\left((f * \phi)(x+h) - (f * \phi)(x)\right) \\ &= \frac{1}{h}\left(\int \phi(x+h-y)f(y)dy - \int \phi(x-y))f(y)dy\right) \\ &= \int \frac{1}{h}(\phi(x+h-y) - \phi(x-y))f(y)dy. \end{aligned}$$

Now, collect everything in the integrand except $f(y)$, and call it

$$A_h(y) = \frac{1}{h}(\phi(x+h-y) - \phi(x-y)).$$

We can see that each $A_h(y)$ is bounded by $||\phi'||_\infty$ by the mean value theorem (why?). If we fix x and pick a small $h > 0$, then the support of A_h is contained in a set E, which is a small neighborhood of the support of ϕ (again, why?).

Define the function $B(y) = ||\phi'||_\infty |f| \chi_E(y)$. It is easy to see that $B \in L^1$ (why?). Now, we have that $|A_h(y)f(y)| \leq |B(y)|$ for every small positive h, so we can apply the dominated convergence theorem to get

$$\lim_{h\to 0} d(h) = \int f(y) \lim_{h \to 0} A_h(y)dy = \int f(y)\phi'(x-y)dy.$$

Now notice that the left-hand side is $(\phi * f)'$ and the right-hand side is $\phi' * f$, so we are done. □

Exercise 6.4. Use Lemma 6.7 to show that $f, g \in \mathcal{S}$ implies $f * g \in \mathcal{S}$. Hint: Use the fact that $(1 + |x|) \leq (1 + |y|)(1 + |x - y|)$.

Lemma 6.8. For two functions, $f, g \in L^1$, $\widehat{f * g} = \hat{f}\hat{g}$.

Proof. Suppose that $f, g \in L^1$, then we write

$$\begin{aligned}
\widehat{f*g}(\xi) &= \int e^{-2\pi i x\cdot\xi}(f*g)(x)dx \\
&= \int e^{-2\pi i x\cdot\xi}\int f(x-y)g(y)dydx \\
&= \int\int e^{-2\pi i x\cdot\xi} f(x-y)g(y)dydx \\
&= \int\int e^{-2\pi i x\cdot\xi} f(x-y)g(y)dxdy \qquad (6.2)\\
&= \int\int e^{-2\pi i(x+(y-y))\cdot\xi} f(x-y)g(y)dxdy \\
&= \int\int e^{-2\pi i(x-y)\cdot\xi} f(x-y)e^{-2\pi i y\cdot\xi}g(y)dxdy \\
&= \int e^{-2\pi i y\cdot\xi}g(y)\int e^{-2\pi i(x-y)\cdot\xi} f(x-y)dxdy \\
&= \int e^{-2\pi i y\cdot\xi}g(y)\int e^{-2\pi i(z)\cdot\xi} f(z)dzdy \qquad (6.3)\\
&= \int e^{-2\pi i y\cdot\xi}g(y)\hat{f}(\xi)dy \\
&= \hat{f}(\xi)\int e^{-2\pi i y\cdot\xi}g(y)dy \\
&= \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

We used Fubini in (6.2) and a change of variables, $z = x - y$, in (6.3). □

Theorem 6.9 (Young). *If $f \in L^p, g \in L^q$, and*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

with $1 \leq p, q, r \leq \infty$, then

$$\|f*g\|_r \leq \|f\|_p\|g\|_q.$$

6.4 Approximate identities

So, it should be apparent from the prequel that convolution with nice things makes things nice. For example, if I have a relatively nasty function f, which is merely L^1_{loc}, I can convolve it with some function $\phi \in C_0^\infty$, to get a different function which has characteristics of both, but is now C^∞. So, it stands to reason that if we could find a relatively nice ϕ, we could "clean up" a less-than-perfect function f and do some more detailed analysis on it– or rather something almost indistinguishable from it. Recall that definition of a *dilation* of a function $f : \mathbb{R}^d \to \mathbb{R}$ is $f^\epsilon = \epsilon^{-d} f(\epsilon^{-1}x)$.

Definition 6.10. A family of functions $\{\phi^\epsilon\}$ (or sometimes just ϕ itself) is called an *approximate identity* if $\phi \in \mathcal{S}$ and $\int \phi = 1$.

This is because we can take an arbitrary $f \in L^1_{loc}$, convolve it with ϕ^ϵ, and get something that is almost exactly like f, but is now C^∞. The following shows that we can also recover f exactly if we let $\epsilon \to 0$.

Lemma 6.11. Let ϕ be an approximate identity. If f is continuous, and f goes to zero at infinity, then $f * \phi^\epsilon \to f$ uniformly as $\epsilon \to 0$. Similarly, if $f \in L^p, 1 \leq p < \infty$ then we get the same convergence in L^p.

Viewing the integration of a function against a measure as a linear transformation shows us that the Fourier transform has the following duality property.

Exercise 6.5 (Duality)**.** Show that for $f, g \in L^1$, that $\int f\hat{g} = \int \hat{f} g$. Hint: Prove that $\int \hat{\nu} d\mu = \int \hat{\mu} d\nu$.

Now we turn to Fourier inversion. This is the inverse of the Fourier transform. Sometimes, we write $f^\vee$

Theorem 6.12. *Suppose that $f, \hat{f} \in L^1$, then for a.e. x,*

$$f(x) = \int e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi.$$

Proof. Consider the following:

$$I_\epsilon(x) = \int \hat{f} e^{-\pi\epsilon^2|\xi|^2} e^{2\pi i \xi \cdot x} d\xi.$$

Now, if we take the limit as $\epsilon \to 0$, we can see that, by the dominated convergence theorem and the fact that $\hat{f} \in L^1$,

$$I_\epsilon(x) \to \int \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

On the other hand, if we let g denote the integrand of the Fourier transform of a type of Gaussian,

$$g(\xi) = e^{-\pi\epsilon^2|\xi|^2} e^{2\pi i \xi \cdot x},$$

then, by appealing to the previous exercise, we get that

$$I_\epsilon(x) = \int f(y)\hat{g}(y)dy.$$

Now, notice that $\hat{g}(y) = \Gamma^\epsilon(x-y)$. We can see that $I_\epsilon = f * \Gamma^\epsilon$. Notice that Γ^ϵ is an approximate identity, so by Lemma 6.11, we get that $I_\epsilon \to f$ as $\epsilon \to 0$. □

Exercise: Show that for a.e. x, $\widehat{\hat{f}}(x) = f(-x)$ for suitable f.

Theorem 6.13 (Parseval)**.** *If $f, g \in \mathcal{S}$, then*

$$\int \hat{f}\overline{\hat{g}} = \int f\overline{g}.$$

Proof. By the definition of the Fourier transform, Fubini, and Theorem 6.12,

$$\begin{aligned}
\int \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi &= \int \left(\int e^{-2\pi i x\cdot\xi} f(x)dx \right) \overline{\hat{g}(\xi)}d\xi \\
&= \int f(x) \left(\int e^{-2\pi i x\cdot\xi}\overline{\hat{g}(\xi)}d\xi \right) dx \\
&= \int f(x)\overline{\left(\int e^{2\pi i x\cdot\xi}\hat{g}(\xi)d\xi \right)}dx \\
&= \int f(x)\overline{g(x)}dx.
\end{aligned}$$

□

Corollary 6.14 (Plancherel)**.** If $f \in \mathcal{S}$, then $\|f\|_2 = \|\hat{f}\|_2$.