3 Function Spaces

3.1 Norms and inner products

Definition 3.1. A *norm* is a function from some space, X to $[0, \infty)$, sending $x \mapsto \|x\|$, and satisfying the following properties for all $x, y \in X$:

- i) $\|x\| = 0$ iff $x = 0$,
- ii) $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \in \mathbb{C}$,
- iii) $\|x + y\| \leq \|x\| + \|y\|$.

Definition 3.2. For $0 < p < \infty$, the space of functions, f, for which $\int |f|^p < \infty$ is called the *L^p-space*. It's norm is

$$\|f\|_{L^p} = \left(\int |f|^p \right)^{\frac{1}{p}}.$$

We often write $\|f\|_{L^p}$ as $\|f\|_p$ for short. For the space L^∞, we use the norm

$$\|f\|_\infty = \lim_{n \to \infty} \|f\|_p = \inf \{B > 0 : \mu(\{x : |f(x)| > B\}) = 0\}.$$

It just so happens that L^p spaces are Banach spaces for $1 \leq p \leq \infty$. In the case of L^2, we have a Hilbert space, and we can have the following.

Definition 3.3. Let X be a space. An *inner product* or *scalar product* is a map from $X \times X$ to $\mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle$ satisfying, for all $x, y, z \in X, \alpha, \beta \in \mathbb{C}$:

- i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
- ii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$,
- iii) $\langle x, x \rangle \in (0, \infty)$ when $x \neq 0$.

Unless otherwise stated, we assume that $\|x\|^2 = \langle x, x \rangle$.

Definition 3.4. A space is said to be *compact* if every one of its open covers has a finite subcover. That is, we call X compact, if for every collection of open subsets of X, $\{U_\alpha\}_{\alpha \in A}$ such that

$$X \subset \bigcup_{\alpha \in A} U_\alpha,$$

we have that there exists a finite subcollection, $\{U_\alpha\}_{\alpha \in B}, B \subset A$, such that

$$X \subset \bigcup_{\alpha \in B} U_\alpha.$$

Definition 3.5. A function is said to be *locally integrable*, or in L^1_{loc} if it is integrable on every compact subset.

3.2 Distribution functions

Here, we follow much of the exposition of Loukas Grafakos' wonderful books.

Definition 3.6. If f is a measurable function on X, the *distribution function of f* is

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}).$$

Example 3.7. Let $f : [0,1] \to [0,1]$ be $f : x \mapsto x^2$. Deduce that

$$d_f(\alpha) = 1 - \sqrt{\alpha}.$$

Definition 3.8. Sometimes, we can't quite get a result for an L^p-space, but we can, in some sense, almost get it. The *weak L^p-spaces* are defined by the norm

$$\|f\|_{L^{p,\infty}} = \sup\{\alpha d_f(\alpha)^{\frac{1}{p}} : \alpha > 0\}.$$

Exercise 3.1. Show the following:

- i) $|g| \leq |f|$ a.e. implies $d_g \leq d_f$.
- ii) $d_{\lambda f}(\alpha) = d_f\left(\frac{\alpha}{|\lambda|}\right)$, for $\lambda \in \mathbb{C} \setminus \{0\}$.
- iii) $d_{f+g}(\alpha + \beta) \leq d_f(\alpha) + d_g(\beta)$.
- iv) $d_{fg}(\alpha\beta) \leq d_f(\alpha) + d_g(\beta)$.

Proposition 3.9. Given $f \in L^p, 0 < p < \infty$, we have

$$\|f\|_p^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha.$$

Proof.

$$\begin{aligned}
p\int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha &= p\int_0^\infty \alpha^{p-1}\left(\int_X \chi_{\{|f|>\alpha\}}(x) d\mu(x)\right) d\alpha \\
&= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha \; d\mu(x) \\
&= \int_X |f(x)|^p d\mu(x) = \|f\|_p^p.
\end{aligned}$$

□

Exercise 3.2. Given $f \in L^p, 0 < p < \infty$, show that for an increasing, continuously differentiable function, ϕ, with $\phi(0) = 0$, we have

$$\int_X \phi(|f|) = \int_0^\infty \phi'(\alpha) d_f(\alpha) d\alpha.$$

3.3 Inequalities

Theorem 3.10 (Cauchy-Schwarz).

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Theorem 3.11 (Hölder). *When* $1 \leq p, q \leq \infty$*, and* $\frac{1}{p} + \frac{1}{q} = 1$*, given two measurable functions on a measure space:*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Exercise 3.3. Use Hölder to deduce Cauchy-Schwarz.

Definition 3.12. A function f mapping into the reals is called *convex* if for every $\lambda \in [0, 1]$, we have that for x and y in the domain,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

Theorem 3.13 (Jensen). *Suppose that* f *is convex, and that* g *is real and integrable. Then*

$$f\left(\int g\right) \leq \int f(g).$$

Theorem 3.14 (Arithmetic-Geometric). *Given a sequence,* $\{x_n\}$*, of non-negative real numbers,*

$$\left(\prod_n x_n\right)^{\frac{1}{n}} \leq \frac{1}{n}\left(\sum_n x_n\right).$$

Exercise 3.4. Use Jensen's inequality to deduce the Arithmetic-Geometric inequality.

Theorem 3.15 (Markov, Pigeonhole). *If* f *is real and measurable, then*

$$\mu(\{|f(x)| \geq t\}) \leq \frac{1}{t}\int |f| d\mu.$$

Proof. $\int |f| d\mu \geq \int t\chi_{\{f \geq t\}} d\mu = t\mu(\{|f(x)| \geq t\})$. □

Theorem 3.16 (Chebychev). *If* f *is real and measurable, then*

$$\mu(\{|f(x)| \geq t\}) \leq \frac{1}{t^2}\int |f|^2 d\mu.$$

Exercise 3.5. Modify the proof of Markov's inequality to prove If f is non-negative and measurable, and g is non-negative, and non-decreasing, then

$$\mu(\{|f(x)| \geq t\}) \leq \frac{1}{g(t)}\int g \circ f d\mu,$$

and from this deduce Chebychev's inequality.

Theorem 3.17 (Minkowski). *For* $1 \leq p \leq \infty$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.