4 Interpolation

Many of these arguments follow Loukas Grafakos' wonderful exposition in his Fourier Analysis books.

4.1 Marcinkiewicz

Definition 4.1. Given measure spaces (X, μ) and (Y, ν), an operator, $T : X \to Y$, is said to be *linear* if for all $f, g \in X, \lambda \in \mathbb{C}$,

$$T(f+g) = T(f) + T(g) \text{ and } T(\lambda f) = \lambda T(f).$$

The operator is called *quasi-linear* if for some $K > 0$, we have that for all $f, g \in X, \lambda \in \mathbb{C}$,

$$|T(f+g)| \leq K(|T(f)| + |T(g)|) \text{ and } |T(\lambda f)| = |\lambda||T(f)|.$$

The operator is called *sublinear* if it is quasi-linear with $K = 1$.

If $\|T(f)\|_q \lesssim \|f\|_p$, we say that T *maps* L^p *to* L^q.

Exercise 4.1. Show that for $x, y > 0$, $xy \leq n$ implies that $\min\{x, y\} \leq n^{\frac{1}{2}}$.

Theorem 4.2 (Marcinkiewicz)**.** *Let (X, μ) and (Y, ν) be two measure spaces, and let $0 < p_0 < p_1 \leq \infty$. Let a sublinear operator T take functions from $L^{p_0}(X) + L^{p_1}(X)$ to measurable functions on Y. Suppose that there exist constants $A_0, A_1 > 0$ such that*

$$\|T(f)\|_{p_0,\infty} \leq A_0 \|f\|_{p_0}, \textit{ for every } f \in L^{p_0}, \textit{ and}$$

$$\|T(f)\|_{p_1,\infty} \leq A_1 \|f\|_{p_1}, \textit{ for every } f \in L^{p_1}.$$

Then for all $p \in (p_0, p_1)$, we have an explicit constant A for which

$$\|T(f)\|_p \leq A\|f\|_p.$$

Proof. Suppose that $p_1 < \infty$. Fix $f \in L^p(X)$ and $\alpha > 0$. Now, we will decompose f into two parts, $f = f_0^\alpha + f_1^\alpha$, where $f_j^\alpha \in L^{p_j}$. This is done by letting f_0^α coincide with f wherever $|f| > \delta\alpha$, (for some $\delta > 0$, to be chosen later) and setting it zero everywhere else. The same is done for f_1^α, but taking the rest of the support of f. Now, we notice that the big part of f, which lives in f_0^α, will be in L^{p_0}, and the little part, f_1^α will be in L^{p_1}. Don't believe me?

$$\|f_0^\alpha\|_{p_0}^{p_0} = \int_{\{|f|>\delta\alpha\}} |f|^p |f|^{p_0-p} d\mu(x) \leq (\delta\alpha)^{p_0-p} \|f\|_p^p.$$

Exercise 4.2. Do this for f_1^α.

Now, since T is sublinear, we have that $|T(f)| \leq |T(f_0^\alpha)| + |T(f_1^\alpha)|$. So we have that

$$\{x : |T(f)(x)| > \alpha\} \subset \left\{x : |T(f_0^\alpha)(x)| > \frac{\alpha}{2}\right\} \cup \left\{x : |T(f_0^\alpha)(x)| > \frac{\alpha}{2}\right\},$$

which implies the estimate of the distribution functions

$$d_{T(f)}(\alpha) \leq d_{T(f_0^\alpha)}\left(\frac{\alpha}{2}\right) + d_{T(f_1^\alpha)}\left(\frac{\alpha}{2}\right). \tag{4.1}$$

Now, take a moment to see that

$$\begin{aligned} d_{T(f_0^\alpha)}\left(\frac{\alpha}{2}\right) &= \left(\frac{\alpha/2}{\alpha/2}\right)^{p_0} d_{T(f_0^\alpha)}\left(\frac{\alpha}{2}\right) \\ &\leq (\alpha/2)^{-p_0} \sup_{\alpha/2>0} \left\{(\alpha/2)^{p_0} d_{T(f_0^\alpha}(\alpha/2)\right\} \\ &= (\alpha/2)^{-p_0} \|T(f_0^\alpha)\|_{p_0,\infty} \leq (\alpha/2)^{-p_0} \|T(f)\|_{p_0,\infty}. \end{aligned}$$

Now we combine this (and a similar calculation for f_1^α) with the assumption that $\|T(f)\|_{p_0,\infty} \leq A_0 \|f\|_{p_0}$ and (4.1) to get

$$d_{T(f)}(\alpha) \leq \frac{A_0^{p_0}}{(\alpha/2)^{p_0}} \int_{\{|f|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x) + \frac{A_1^{p_1}}{(\alpha/2)^{p_1}} \int_{\{|f|\leq\delta\alpha\}} |f(x)|^{p_1} d\mu(x).$$

Now, using Proposition 3.9 from before, we get

$$\begin{aligned} \|T(f)\|_p^p &\leq p(2A_0)^{p_0} \int_0^\infty \alpha^{p-1} \alpha^{-p_0} \int_{\{|f|>\delta\alpha\}} |f(x)|^{p_0} d\mu(x) d\alpha + \\ &\quad + p(2A_1)^{p_1} \int_0^\infty \alpha^{p-1} \alpha^{-p_1} \int_{\{|f|\leq\delta\alpha\}} |f(x)|^{p_1} d\mu(x) d\alpha \\ &\leq p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{\delta^{-1}|f(x)|} \alpha^{p-1-p_0} d\alpha d\mu(x) + \\ &\quad + p(2A_1)^{p_1} \int_X |f(x)|^{p_1} \int_{\delta^{-1}|f(x)|}^\infty \alpha^{p-1-p_1} d\alpha d\mu(x) \\ &= \frac{p(2A_0)^{p_0}}{(p-p_0)\delta^{p-p_0}} \int_X |f(x)|^{p_0} |f(x)|^{p-p_0} d\mu(x) + \\ &\quad + \frac{p(2A_1)^{p_1}}{(p_1-p)\delta^{p-p_1}} \int_X |f(x)|^{p_1} |f(x)|^{p-p_1} d\mu(x) \\ &= p\left(\frac{p(2A_0)^{p_0}}{(p-p_0)\delta^{p-p_0}} + \frac{p(2A_1)^{p_1}}{(p_1-p)\delta^{p-p_1}}\right) \|f\|_p^p. \end{aligned}$$

Now, pick δ such that

$$\frac{p(2A_0)^{p_0}}{(p-p_0)\delta^{p-p_0}} = \frac{p(2A_1)^{p_1}}{(p_1-p)\delta^{p-p_1}}.$$

The case $p_1 = \infty$ is left to the interested reader. $\square$

4.2 Riesz-Thorin

Lemma 4.3 (Hadamard's three lines lemma)**.** Let F be analytic in the open strip, $S = \{z \in \mathbb{C} : 0 < Re\ z < 1\}$, continuous and bounded on its closure, $\overline{S}$, such that $|F(z)| \leq B_0 < \infty$ when $Re\ z = 0$, and $|F(z)| \leq B_1 < \infty$ when $Re\ z = 1$. Then for any $\theta \in [0,1]$, for any z with $Re\ z = \theta$,

$$|F(z)| \leq B_0^{1-\theta} B_1^{\theta}.$$

Proof. Define analytic functions

$$G(z) = F(z)(B_0^{1-z}B_1^z)^{-1} \quad \text{and} \quad G_n(z) = G(z)e^{(z^2-1)/n}.$$

Since F is bounded on the closed unit strip, and $B_0^{1-z}B_1^z$ is bounded from below, we know that G is bounded by some constant $M < \infty$ on the closed strip. Also, $G \leq 1$ the boundary. Now,

$$|G_n(x+iy)| \leq Me^{-y^2/n}e^{x^2-1/n} \leq Me^{-y^2/n},$$

so we know that G_n converges to zero uniformly in the closed strip as y approaches $\pm\infty$. Now, pick some y_0 such that for all $|y| \geq |y_0|$, we can bound $|G_n| \leq 1$, uniformly in x in the closed strip. By the maximum modulus principle, we get that $|G_n| \leq 1$ in the rectangle $[0,1] \times [-|y_0|, |y_0|]$. So, we get that $|G_n| \leq 1$ everywhere in the closed strip. Now, if we let $n \to \infty$, we get $|G| \leq 1$ also in the closed strip. $\square$

Theorem 4.4 (Riesz-Thorin)**.** *Let (X, μ) and (Y, ν) be two measure spaces, and let $0 < p_0, p_1, q_0, q_1 \leq \infty$. Let T be a linear operator mapping simple functions on X to measurable functions on Y. Suppose that there exist constants $A_0, A_1 > 0$ such that*

$$\|T(f)\|_{q_0} \leq M_0\|f\|_{p_0}, \text{ and}$$

$$\|T(f)\|_{p_1} \leq M_1\|f\|_{p_1}, \text{ for every simple } f \in X.$$

Then for all $\theta \in (0,1)$, we have

$$\|T(f)\|_q \leq M_0^{1-\theta}M_1^{\theta}\|f\|_p$$

for all simple functions F on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

By density, T has a unique extension as a bounded operator from $L^p(X,\mu)$ to $L^q(Y,\nu)$ for all suitable p and q.

Proof. Define f to be a simple function on X,

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \chi_{A_k},$$

where $a_k > 0$, $\alpha_k \in \mathbb{R}$, and A_k are disjoint subsets of X of finite measure. Now, we'd like to get a handle on $\|T(f)\|_{L^q(Y)}$.

$$\|T(f)\|_{L^q(Y)} = \sup \left| \int_Y T(f)(x) g(x) d\nu(x) \right|,$$

where the supremum is taken over simple functions g on Y such that

$$\|g\|_{L_{q'}(Y)} \leq 1.$$

To continue, we can write g as

$$g = \sum_{j=1}^{n} b_j e^{i\beta_j} \chi_{B_j},$$

where $b_j > 0$, $\beta_j \in \mathbb{R}$, and B_k are disjoint subsets of Y of finite measure. Now, we define two functions on $\overline{S} = \{z \in \mathbb{C} : 0 \leq z \leq 1\}$,

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1} z \quad \text{and} \quad Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1} z.$$

Now, we define a two functions like f and g above,

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k} \quad \text{and} \quad g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j}.$$

Finally, we can define a function F on $\overline{S}$,

$$F(z) = \int_Y T(f_z)(x) g_z(x) d\nu(x).$$

Now, by linearity, we get that

$$F(z) = \sum_{k=1}^{m}\sum_{j=1}^{n} a_k^{P(z)} b_j^{Q(z)} e^{i(\alpha_k+\beta_j)} \int_Y T(\chi_{A_k})(x)\chi_{B_j}(x)d\nu(x).$$

From this, we get that F is analytic in z because the coefficients, $a_k, b_j > 0$.

Now, consider some z whose real part is zero. Recall that

$$\left|a_k^{P(z)}\right| = a_k^{\frac{p}{p_0}}.$$

Now, as the A_k are disjoint, we get that

$$\|f_z\|_{p_0}^{p_0} = \|f\|_p^p.$$

Similarly, we get that

$$\|g_z\|_{q_0'}^{q_0'} = \|g\|_{q'}^{q'}.$$

Now, we apply Hölder to get

$$|F(z)| = |\int_Y T(f_z)(x)g_z(x)d\nu(x)| \leq \|T(f_z)\|_{q_0}\|g_z\|_{q_0'},$$

which, by assumption is bounded above by

$$\leq M_0\|f_z\|_{p_0}\|g_z\|_{q_0'} = M_0\|f_z\|_p^{\frac{p}{p_0}}\|g_z\|_{q'}^{\frac{q'}{q_0'}}.$$

If we now consider a z whose real part is 1, we will get

$$\|f_z\|_{p_1}^{p_1} = \|f\|_p^p, \quad \text{and} \quad \|g_z\|_{q_1'}^{q_1'} = \|g\|_{q'}^{q'},$$

by the definition of f_z and the fact that the magnitude of a purely imaginary exponential is 1. This will yield

$$|F(z)| \leq M_1\|f_z\|_p^{\frac{p}{p_1}}\|g_z\|_{q'}^{\frac{q'}{q_1'}}.$$

Now, F is analytic in the open strip, S, and continuous on its closure. We also know that F is bounded on the closure of the strip. So, by Hadamard's three lines lemma, we get that, when the real part of z is θ,

$$|F(z)| \leq \left(M_0\|f_z\|_p^{\frac{p}{p_0}}\|g_z\|_{q'}^{\frac{q'}{q_0'}}\right)^{1-\theta}\left(M_1\|f_z\|_p^{\frac{p}{p_1}}\|g_z\|_{q'}^{\frac{q'}{q_1'}}\right)^{\theta} = M_0^{1-\theta}M_1^{\theta}\|f\|_p\|g\|_{q'}.$$

We can see that $P(\theta) = Q(\theta) = 1$, so we get that

$$F(\theta) = \int_Y T(f)gd\nu.$$

We finish by taking the supremum over all simple functions on Y with $L^{q'}$ norm ≤ 1, and using density to extend T.

□