1 Preliminaries

If A and B are sets, then A^c is the complement of A (with respect to some universe), $A \cup B$ is their union, $A \cap B$ is their intersection, $A \setminus B$ is $A \cap B^c$, and the power set of A is $\mathcal{P}(A) := \{E : E \subset A\}$. Often, we index a family of sets E_j. We write a sequence of elements as $\{x_j\}$. If two quantities, X and Y, vary with respect to some parameter, n, we say $X \lesssim Y$ if $X \leq CY$, for some constant, $C > 0$, which does not depend on n. We write $X \approx Y$ when $X \lesssim Y$ and $Y \lesssim X$.

Definition 1.1.

$$\limsup E_j = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j \text{ and } \liminf E_j = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} E_j.$$

So the limit supremum of a family of sets is the set of elements in infinitely many of the sets, and the limit infimum is the set of elements in all but finitely many sets. Check that for a family of sets, E_j, lim inf $E_j \subset$ lim sup E_j.

Example 1.2. Define E_j to be the interval $[-\frac{1}{j}, 10 + (-1)^j]$. Then

$$\limsup E_j = [0, 11] \text{ and } \liminf E_j = [0, 9].$$

Definition 1.3. A *metric* on a set X is a function $\rho : X \times X \to [0, \infty)$ such that the following hold:

i) $\rho(x, y) = 0$ iff $x = y$, (coincidence);

ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, (symmetry);

iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, (triangle inequality).

A *metric space* is a pair (X, ρ). When the metric ρ is clear from context, we write $|x - y|$.

Definition 1.4. Given a subset, $E \subset \mathbb{R}$, the greatest lower bound, or *infimum* is defined to be the greatest number that is less than or equal to every number in E. That is,

$$\inf E = \max\{x : x \leq y, \forall y \in E\}.$$

Similaraly, the least bound, or *supremum* is defined to be the least number that is greater than or equal to every number in E. That is,

$$\sup E = \min\{x : x \geq y, \forall y \in E\}.$$

Definition 1.5. Given a metric and two sets, E_1 and E_2, the distance from E_1 to E_2 is defined to be $|E_1 - E_2| = \inf \{|x - y| : x \in E_1, y \in E_2\}$. The *diameter* of E_1 is $\text{diam}(E_1) = \sup\{|x - y| : x, y \in E_1\}$.

Definition 1.6. Given a metric space, (X, ρ), and sequence of elements of the space, $\{x_j\}$ we say that the some element $L \in X$ is the *limit* of the sequence if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|x_n - L| \leq \epsilon.$$

Definition 1.7. Given two metric spaces, A and B, a function $f : A \to B$ is said to be *continuous* iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that for x, y in the domain of f, we have that $|x - y| \leq \delta$ implies that $|f(x) - f(y)| \leq \epsilon$.

Definition 1.8. A *topology* on a set X is a subset of the power set of X, $\mathcal{T} \subset \mathcal{P}(X)$, such that the following hold:

i) $\varnothing, X \in \mathcal{T}$

ii) Any (possibly infinite) union of elements of $\mathcal{T}$ is in $\mathcal{T}$.

iii) An intersection of any finite number of elements of $\mathcal{T}$ is in $\mathcal{T}$.

We call the sets in $\mathcal{T}$ the *open sets*. A subset of X is called *closed* if it is the complement of an open set. If $E \subset X$, then the union of all subsets of E is called the *interior* of E, and often written E°. The intersection of all closed sets containing E is called the *closure* of E, and often denoted $\overline{E}$. The set $\overline{E} \setminus E^\circ$ is called the *boundary* of E, and often denoted ∂E.

Definition 1.9. The *standard topology* on $\mathbb{R}$ is generated by unions and finite intersections of intervals of the form (x, y), along with the empty set and $\mathbb{R}$ itself.

Exercise 1.1. Show that a function, $f : \mathbb{R} \to \mathbb{R}$ is continuous if, in the standard topology, the pre-image of every open set is open.

Exercise 1.2. Consider the set of polynomials in $\mathbb{R}^d$, denoted $\mathbb{R}^d[x]$. For any subset $S \subset \mathbb{R}^d[x]$, we can construct the *algebraic set* $Z(S)$ as follows:

$$Z(S) := \{x \in \mathbb{R}^d : f(x) = 0, \forall f \in S\}.$$

The *Zariski topology* on $\mathbb{R}^d$ is defined as the complements of the $Z(S)$ where S ranges through all possible subsets of $\mathbb{R}^d[x]$. Show that the Zariski topology is indeed a topology.