Harmonic Analysis and Geometric Combinatorics

Steven Senger

1. Overview

This course will give students an introduction harmonic analysis with emphases on geometric measure theory and combinatorics. The main topics will be Fourier analysis, measure and integration theory, incidence geometry, and finite fields. The motivating theme will be the study of a family of geometric problems related to the Erdős and Falconer distance problems, such as the Kakeya problem, and the sums and products problem. There are two goals with this class. The first is to give students familiarity with the basic tools of measure theory, harmonic analysis, geometric combinatorics, and finite fields. The second goal is to illustrate how seemingly disparate areas of mathematics interact, and how often times a viewpoint which did not seem obvious can be fruitful. The content will follow three main themes: continuous theory, discrete theory, and finite field theory. By the end of the course, all three will be tightly woven together in a unified set of tools and techniques.

The first section (Measure Theory) will provide a basic framework for the remainder of the course. We give enough background for the students to be comfortable with measures and Lebesgue integration, which will be crucial for what follows.

The second section (Harmonic Analysis) begins with a broad view of the fundamental concepts of harmonic analysis. We then address some elementary questions that arise naturally in Fourier analysis. It will quickly become apparent that a geometric perspective is extremely useful for studying these problems, and lead into study of the appropriate geometric underpinnings.

The third section (Discrete Geometry) will give a formal introduction to the types of geometric tools called for by the previous section. To accomplish this, we introduce fundamental techniques from discrete geometry through incremental results toward the Erdős distance problem. Along the way, students will see how to apply geometric reasoning to analytic problems. However, students will also see the analytic ideas (dyadic decomposition, Cauchy-Schwarz and Jensen inequalities, etc.) employed in order to finish many of the basic arguments. In particular, we will show how Falconer's estimate relates to the unit distance problem (a stronger Erdős distance problem).

The fourth section (Finite Fields) will give a basic refresher of finite fields, and dive right into character sums and exponential sums, which can be thought of as discrete versions of the Fourier Transform. It is not very surprising that arguments inspired by the discrete geometry in the previous section can transfer to the finite field setting. What is surprising is how geometric averages in finite fields can yield results based on density which look just like the corresponding analytic results based on Hausdorff dimension. We will discuss this and more, and in so doing, generate a partial dictionary on how to translate ideas back and forth between various settings.

The fifth section (Tying it all together) is the capstone of the course. Here, we present a few of the easily accessible results that have come directly from applying ideas in one of these areas to another. One example, s-adaptability, has provided a conversion mechanism between analytic results and discrete geometry, which has benefitted both areas. To show real world applications, we discuss robust quantized-space results, which directly apply to approximation and computation, as well as a swarming algorithm reliability scheme.

2. Pre-requisites

Math 451, Math 401, and Math 210 (or equivalents) are strongly recommended.

3. Assessment

Student performance will be assessed in the following way:

- 25% - regular ten-minute quizzes
- 25% - Exam 1
- 25% - in-class presentation (one or two per student, depending on class size)
- 25% - Exam 2

4. Course Materials

The following course materials will be suggested.

- *Lecture Notes on Harmonic Analysis* - Tom Wolff, AMS, University Lecture Series, vol. 29 (a preliminary version is available online for free)
- *The Erdős Distance Problem* - Julia Garibaldi, Alex Iosevich, Steven Senger, AMS Student Mathematical Library, vol. 56
- Several research publications.

5. Lecture Schedule

5.1. Weeks 1-2: Measure Theory.

- Topology, set theory
- Lebesgue measure and integration
- Dominated Convergence Theorem, L^p spaces
- Cauchy-Schwarz, Hölder, and Jensen Inequalities

5.2. Weeks 3-6: Harmonic Analysis.

- Covering Lemmas, Maximal Functions
- Approximate identities, distributions, and dimension
- Fourier Transform, Interpolation, (Hilbert Transform?)
- Riesz Potential, Frostman measure
- Stationary Phase, Falconer's estimate
- Restriction phenomena, Kakeya problem

5.3. Weeks 8-10: Discrete Geometry.

- Erdős distance problem and $n^{\frac{1}{2}}$ argument, Moser's argument
- Graph theory and incidence geometry, Székely's argument
- K-circles, ℓ^p-circles, and Valtr's construction
- Dot products and Poincaré half-plane
- Solymosi-Toth argument
- Information theory and entropy, The Katz-Tardos argument

5.4. Weeks 11-12: Finite Fields.

- Finite Fields, Character sums
- Orthogonal vectors, Gauss and Kloosterman sums
- Finite field Erdős distance problem
- Hinges, triangles, and the finite field Kakeya problem

5.5. Week 13: Tying it all together.

- s-adaptability, Multi-scale analysis, Continuous and finite field Valtr
- Swarm algorithm measurement, Sums and products, Guth-Katz realization of the Elekes-Sharir Framework

6. Additional Policy

Academic honesty will be strictly enforced. You may not cheat or copy work. Do your own work. Hand in your own work. Remember that allowing someone else to copy your work is considered cheating. Students are encouraged to work together outside of class. Please review the Student Code of Conduct if you have any confusion as to what is to be expected. Violations will be handled by the Office of Student Conduct.

The Americans with Disabilities Act (ADA) is a federal anti-discrimination stat- ute that provides comprehensive civil rights protection for persons with disabilities. Among other things, this legislation requires that all students with disabili- ties be guaranteed a learning environment that provides for reasonable accommodation of their disabilities. If you believe you have a disability requiring an accommodation, please contact the campus disability services office, www.udel.edu/DSS/.