Elementary proof of Elekes' sums and products result
Lecture notes by Steven Senger, prepared for Oct. 15, 2010.

The first four results are standard and elementary.

Proposition 1 (Euler's formula)**.** *Suppose G is a connected planar graph with v vertices, e edges, and f faces, then $v + f = e + 2$.*

Proof. Start with a graph consisting of only one vertex. Construct the graph G by adding one edge at a time. Notice that any time you add an edge, you either add a vertex or a face. □

Proposition 2. *Suppose G is a connected planar graph with $e \geq 2$ edges, and f faces, then $3f \leq 2e$.*

Proof. Draw the graph. For each edge in the graph, draw a small segment normal to the edge extending only on one side of the edge. Observe that there are two "sides" of each edge, the side with a small segment, and the side without. Notice that each face needs **at least** three sides, and each edge has two sides. □

Combining Propositions 1 and 2 yields the following corollary.

Corollary 1. *Suppose G is a planar graph with $e \geq 2$ edges, and v vertices, then $e - 3v + 6 \leq 0$.*

Now, the key to the main lemma.

Proposition 3. *Suppose G is a graph (possibly nonplanar) with $e \geq 2$ edges, and f faces, then the minimum number of crosssings under any redrawing of G is*

$$cr(G) \geq e - 3v + 6. \tag{1}$$

Proof. If the graph has no crossings, then we are done by Corollary 1. If not, get a bag, removee one edge which crosses another edge from the graph and toss it in your bag. Keep pulling edges out until theere are no more crossings. Count the number of edges you have in your bag. Each edge you took out of the graph removed at least one crossing, so, count the number of edges in your bag. □

The next two proofs are from Székely's *Crossing numbers and hard Erdős problems in discrete geometry.* They were both initially shown by other people. See the references in Székely's paper.

Lemma 1 (Crossing Number Lemma)**.** *Suppose G is a graph with e edges, v vertices, and $e \geq 4v$, then*

$$cr(G) \gtrsim \frac{e^3}{v^2}.$$

Proof. Let H be a random induced subgraph of G, where each vertex is chosen with probability $p = \frac{4v}{e}$. Now, write (1) for H, and take expected values of both sides.

$$\mathbb{E}(cr(H)) \leq \mathbb{E}(e_H) - 3\mathbb{E}(v_H) + 6 \tag{2}$$

Verify that $\mathbb{E}(v_H) = pv$, $\mathbb{E}(e_H) = p^2 e$, and $\mathbb{E}(cr(H)) \leq p^4(cr(G))$, and (2) becomes

$$p^4 cr(G) \gtrsim p^2 e - 3pv + 6,$$

which yields the claimed estimate. □

Theorem 1 (Szemerédi-Trotter, *Extremal problems in discrete geometry*)**.** *Given a collection of n points and m lines in the plane, the number of incidences of points and lines is*

$$I \lesssim (nm)^{\frac{2}{3}} + n + m.$$

Proof. Construct a graph, G, by letting the vertices be the n points, and the edges be the $I - m$ segments connecting the consecutive points on a given line. So $v = n$, and $e = I - m$. Notice that the two distinct lines can cross each other only once, so number of crossings in G must be less than $\binom{m}{2} \approx m^2$. If $e \leq 4v$, then $I \leq 4n + m$. If $e > 4v$, then we can apply Lemma 1, and we get that $I \lesssim (nm)^{\frac{2}{3}} + m$, and we are done. □

Theorem 2 (Elekes *On the number of sums and products*)**.** *Given a set, $\mathcal{A}$, of N numbers,*

$$\max\{|\mathcal{A} \cdot \mathcal{A}|, |\mathcal{A} + \mathcal{A}|\} \gtrsim N^{\frac{5}{4}}.$$

Proof. We will construct a set of points and lines. The points will be $\mathcal{P} = \{(a_i + a_j, a_k \cdot a_l) : a_i, a_j, a_k, a_l \in \mathcal{A}\}$, and the lines will be $\mathcal{L} = \{y = a_s \cdot (x - a_t) : a_s, a_t \in \mathcal{A}\}$. There are $|\mathcal{A} + \mathcal{A}| \cdot |\mathcal{A} \cdot \mathcal{A}|$ points, and N^2 lines. Notice that each line is coincident to at least N points. Apply Theorem 1, and see that

$$N^3 = N \cdot |\mathcal{L}| \lesssim I \lesssim \left(|\mathcal{A} + \mathcal{A}| \cdot |\mathcal{A} \cdot \mathcal{A}| \cdot N^2\right)^{\frac{2}{3}} + |\mathcal{A} + \mathcal{A}| \cdot |\mathcal{A} \cdot \mathcal{A}| + N^2.$$

So $|\mathcal{A} + \mathcal{A}| \cdot |\mathcal{A} \cdot \mathcal{A}| \gtrsim N^{\frac{5}{2}}$, and the result follows. □