Genera of Subgroup Intersection Graphs

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A Recap on Groups

- **What is a Group again?**
  - The **order** of a group $G$ is the number of elements in $G$, denoted $|G|$.
  - The **order** of an element $g \in G$ is the smallest positive integer $n$ such that $g^n = 1$ in $G$.

- **Subgroups of $G$**

- **Proper Subgroups of $G$**: Subgroups that are not the entire group $G$. 

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- **A graph** is a collection of vertices $V$ and a collection of edges $E$ which connect the vertices.
- Typically, vertices are represented as points and edges are represented as lines between those points.
- A **Complete Graph on $n$ vertices** $K_n$ is a graph where every vertex is uniquely connected by an edge.
- $K_n$ has $n$ vertices and $\binom{n}{2}$ edges.
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The Genus of a Graph

- Orientable Genus
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Genus Formulas

- For an arbitrary graph $\Gamma$:
  \[ \gamma(\Gamma) \geq \lceil \frac{E}{6} - \frac{V}{2} + 1 \rceil \]
  \[ \tilde{\gamma}(\Gamma) \geq \lceil \frac{E}{3} - V + 2 \rceil \]

- For a complete graph $K_n$:
  \[ \gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \]
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The Hasse Diagram of a given group $G$ is the graph whose vertices are the subgroups of $G$ and whose edges are determined by “Immediate Inclusion”.

Given $H_1, H_2 \leq G$, we connect $H_1$ and $H_2$ with an edge if $H_1 \leq H_2$ and there does not exist a subgroup $H$ such that $H_1 < H < H_2$.

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The Subgroup Intersection Graph (or Intersection Graph) of $G$

- Its vertices are the proper subgroups of $G$, excluding the trivial subgroup $\langle 1 \rangle$.
- Two vertices are connected by an edge iff $H_1 \cap H_2 \neq 1$.
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If we can find one that is larger than genus 1, then we are done!

If the intersection graph is a union of complete subgraphs, we can use the Inclusion-Exclusion principle.

Inclusion-Exclusion Principle: For two sets $A$ and $B$, $|A \cup B| = |A| + |B| - |A \cap B|$.

If we cannot find a subgraph greater than genus 1 we can also explicitly embed the subgroup intersection graph onto a torus or projective plane.

We have stronger tools which will be seen later.
General Strategy

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An Example

- Recall that a **Cyclic Group** is a group that can be generated by a single element.
- We denote a (finite) Cyclic Group of order $n$ by $C_n$.
- The **Fundamental Theorem of Cyclic Groups** states that:
  1. Every subgroup of a cyclic group is cyclic, and
  2. There is a one-to-one correspondence between subgroups of $C_n$ and the divisors of $n$. 
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An Example: $C_{p^3 q}$
A subgroup $H$ of a group $G$ is called a **normal subgroup** of $G$ if $aH = Ha$ for all $a$ in $G$. 
Quotient Groups

- Given a subgroup $H$ of $G$, we consider sets of the form $\{aH \mid a \in G\}$. These sets partition $G$ into $|G|/|H|$ disjoint classes.

- These sets form a group $G/H = \{aH \mid a \in G\}$ under the operation $(aH)(bH) = (ab)H$, which is well-defined when $H$ is normal in $G$.

- The **index** of a subgroup, the number of disjoint sets in the partition, is equal to the order of the $G$ divided by the order of $H$. Intuitively, the index is the “relative size” of $H$ in $G$. 
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Let $G$ be a group and let $p$ be a prime. If $p^k$ divides $|G|$ and $p^{k+1}$ does not divide $|G|$, then any subgroup of $G$ of order $p^k$ is called a **Sylow $p$-subgroup** of $G$.

**Sylow’s First Theorem** states that there must exist at least one subgroup of order $p^k$ if $p^k$ divides $|G|$. 
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Direct and Semi-Direct Products

- Let $G$ and $H$ be groups. We define the direct product of $G$ and $H$ as $H \times G = \{(h, g) | h \in H, g \in G\}$, with the operation of $H \times G$ defined coordinate-wise.

- A semi-direct product is a generalization of the direct product. We say a group $G$ is a semi-direct product of a normal subgroup $H$ and subgroup $K$ denoted $H \rtimes K$ if $H$ and $K$ intersect trivially and $G = HK$. If $H$ and $K$ are both normal, then $G$ is the direct product of $H$ and $K$. 
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An Example: \textit{Fred}_0

\[(C_p \rtimes C_p) \times C_q = \langle a, b, c \mid a^p = b^p = c^q, cac^{-1} = a^i, cb = bc, ab = ba, \text{ord}_p(i) = q \rangle \text{ and } p > q.\]

- Subgroups of order \(p^2\): \langle a, b \rangle
- Subgroups of order \(pq\): \langle a, c \rangle, \langle bc \rangle, \langle b(ac) \rangle, \ldots, \langle b(a^{p-1}c) \rangle
- Subgroups of order \(p\): \langle b \rangle, \langle a \rangle, \langle ab \rangle, \ldots, \langle a^{p-1}b \rangle
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Basic Strategy

- We look at groups whose orders have more and more prime factors until, hopefully, they all have genus greater than 1.
The Three Essential Techniques

If we are trying to show that the genus of the intersection graph of a group $G$ is larger than 1, we can:
1.) Find a subgroup of $G$ whose intersection graph has genus greater than 1.
2.) Find a quotient group $G/N$ with genus greater than 1.
3.) When all else fails, actually find the subgroups of $G$, and draw out all or part of the Hasse diagram!
Our Favorite Tool

- **The Lattice Isomorphism Theorem**: If $N$ is a normal subgroup of a group $G$, then there exists a bijection from the set of all subgroups $H$ of $G$ such that $H$ contains $N$, onto the set of all subgroups of the quotient group $G/N$. The structure of the subgroups of $G/N$ is exactly the same as the structure of the subgroups of $G$ containing $N$, with $N$ collapsed to the identity element.
So what does this mean? It means that the intersection graph of $G/N$ will look exactly the same as the part of the graph of $G$ that’s above the vertex labeled $N$. 

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Example

- \(C_p^3 \times C_p / N \cong C_p^2 \times C_p\) for \(N \cong C_p\), i.e. \(C_p^3 \times C_p\) has \(C_p^2 \times C_p\) as a quotient group.
A Nice Consequence of This Theorem

- Notice that if $G/N$ has $n$ proper subgroups, then the Lattice Isomorphism Theorem gives us a $K_n$ subgraph in the intersection graph of $G$.

- In particular, if $G/N$ has 8 or more proper subgroups, then there will be at least a $K_8$ in the intersection graph of $G$, making its genus greater than 1.
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- In particular, if $G/N$ has 8 or more proper subgroups, then there will be at least a $K_8$ in the intersection graph of $G$, making its genus greater than 1.
We start by dividing finite groups into two categories: **solvable** and **nonsolvable**.
A group $G$ is said to be **solvable** if we can write
\[ \langle 1 \rangle = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G \]
where the order $|H_{i+1}/H_i|$ is prime for all $i$.

A group $G$ is said to be **nonsolvable** if it is not solvable.
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$\langle 1 \rangle = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G$ where the order $|H_{i+1}/H_i|$ is prime for all $i$.

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Fun Facts About Solvable Groups!

- The chain of normal subgroups tells us that the order of $G/H_{n-1}$ is prime. So the order of $H_{n-1}$ has one prime factor less than the order of $G$.
- This gives us a way to induct on the order of $G$. 

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Strategy For Solvable Groups

- For example if $|G| = p^4 q^2 r$, then $G$ has a subgroup of order $p^3 q^2 r$, $p^4 qr$, or $p^4 q^2$.

- If we have shown that all such groups have genus greater than 1, then $G$ is automatically eliminated.
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We can use this fact to find large quotient groups of a given group $G$.

For example, if $|G| = p^2 q^2 r$, then $|N| = p, p^2, q, q^2, r$ or $r^2$.

So $|G/N| = |G|/|N| = p^2 q^2, pq^2 r, q^2 r, p^2 qr, or p^2 r$.

Very few of these groups have fewer than 7 proper subgroups, so we have narrowed down the possibilities substantially.
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### Table of Subgroups

<table>
<thead>
<tr>
<th>Number of Proper Subgroups</th>
<th>Groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_p$</td>
</tr>
<tr>
<td>2</td>
<td>$C_{p^2}$</td>
</tr>
<tr>
<td>3</td>
<td>$C_{pq}, C_{p^3}$</td>
</tr>
<tr>
<td>4</td>
<td>$C_2 \times C_2, C_{p^4}$</td>
</tr>
<tr>
<td>5</td>
<td>$S_3, Q_8, C_3 \times C_3, C_{p^2q}, C_p^5$</td>
</tr>
<tr>
<td>6</td>
<td>$C_{p^6}$</td>
</tr>
<tr>
<td>7</td>
<td>$C_4 \times C_2, D_{10}, C_3 \times C_4, C_5 \times C_5, C_{pqr}, C_{p^3q}, C_{p^7}$</td>
</tr>
</tbody>
</table>
Fun Facts About Solvable Groups!

- And the very best fun fact: Any solvable group whose order has more than 3 distinct prime factors is automatically eliminated; its intersection graph will always have genus greater than 1.
Proof

- Let $G$ be a solvable group of order $p^\alpha q^\beta r^\delta s^\gamma \cdots$. The Sylow Theorems guarantee that $G$ has subgroups $P, Q, R, S$, of orders $p^\alpha, q^\beta, r^\gamma, s^\delta$, respectively.

- Since $G$ is solvable, these form a Sylow Basis; the product of any set of these subgroups is itself a subgroup. For example, $PQ$ and $PQS$ are subgroups of $G$. 

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Genera of Subgroup Intersection Graphs
Proof

- Let $G$ be a solvable group of order $p^\alpha q^\beta r^\delta s^\gamma \cdots$. The Sylow Theorems guarantee that $G$ has subgroups $P, Q, R, S$, of orders $p^\alpha, q^\beta, r^\gamma, s^\delta$, respectively.

- Since $G$ is solvable, these form a Sylow Basis; the product of any set of these subgroups is itself a subgroup. For example, $PQ$ and $PQS$ are subgroups of $G$. 
Proof

- This gives us the following portion of the Hasse diagram of $G$:
Proof

- We see that $Q$ is contained in six other proper subgroups:
Proof

- As is $S$:
Proof

- This will produce two copies of \( K_7 \) meeting at three vertices in the intersection graph of \( G \). We write \( K_7 \vee K_3 K_7 \subseteq \Gamma(G) \).
- This subgraph has \( \binom{7}{2} + \binom{7}{2} - 3 = 39 \) edges and 11 vertices by inclusion-exclusion.
- It has genus at least \( \gamma(K_7 \vee K_3 K_7) \geq \lceil \frac{39}{6} - \frac{11}{2} + 1 \rceil = \lceil \frac{12}{6} \rceil = 2 \). So \( G \) is too big!
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Strategy for Solvable Groups

- Abelian Groups
  - $p$-groups
  - Groups of order $p^2 q$
  - Groups of order $p^\alpha q$
  - Groups of order $p^2 q^2$
  - Groups of order $p^\alpha q^\beta$
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What We Are Working On Now

- We are currently working on solvable groups of order $p^2q^2$ and order $p^2qr$; so far they all have genus greater than 1, so it looks like we have almost reached the end!
Nonsolvable Groups

- Every nonsolvable group contains a minimal simple group as a subquotient.
- In other words, the Hasse diagram of a non-solvable group contains that of a minimal simple group as a sub-lattice.
- There are essentially five possible minimal simple groups: $L_2(2^p)$, $L_2(3^p)$, $L_3(3)$, $L_2(p)$, and $Sz(2^q)$.
- Each of these has a solvable subgroup with genus greater than 1.
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We hope to finish up the remaining solvable groups; in addition, there are a number of other properties of subgroup intersection graphs that could be explored, including:

- Nonorientable Genus
- Hamiltonian Cycles
- Chromatic Number
- Higher genera
- And much, much more!
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