Problem 1: (#3 on page 393 of DeGroot) Suppose that random variables $X_1, \cdots, X_n$ are independently from $N(\theta, \theta^2)$. Denote $\bar{X}$ as the sample mean. Determine $n$ such that $E(|\bar{X} - \theta|) \leq 0.1$.

Problem 2: (#4 on page 393 of DeGroot) Suppose that random variables $X_1, \cdots, X_n$ are independently from $N(\theta, \theta^2)$. Denote $\bar{X}$ as the sample mean. Determine $n$ such that $P(|\bar{X} - \theta| \leq 0.1) \geq 0.95$.

Problem 3: (#1 on page 403 of DeGroot) Suppose that random variables $X_1, \cdots, X_n$ are independently from $N(\mu, \sigma^2)$. Let
\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.
\]
Prove that $\hat{\sigma}^2$ has a gamma distribution with parameters $(n - 1)/2$ and $n/(2\sigma^2)$.

Problem 4: (#7 on page 404 of DeGroot) Suppose that random variables $X_1, \cdots, X_n$ are independently from $N(\mu, \sigma^2)$. Let
\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.
\]
For (a), (b), determine the smallest $n$, such that the following relations holds:

(a) $P\left(\frac{\hat{\sigma}^2}{\sigma^2} \leq 1.5\right) \geq 0.95$

(b) $P\left(|\hat{\sigma}^2 - \sigma^2| \leq 0.5\sigma^2\right) \geq 0.8$

Problem 5: (#3 on page 415 of DeGroot) Suppose that random variables $X_1, \cdots, X_n$ are independently from $N(\mu, \sigma^2)$, with $\mu$ and $\sigma$ unknown. Let $L$ be the length of the confidence interval that can be constructed from the random sample. Find $E(L^2)$ for the following sample size $n$ and confidence level $1 - \alpha$. 


(a) \( n = 5, 1 - \alpha = 0.95 \)  
(b) \( n = 10, 1 - \alpha = 0.95 \)  
(c) \( n = 30, 1 - \alpha = 0.95 \)  
(d) \( n = 8, 1 - \alpha = 0.90 \)  
(e) \( n = 8, 1 - \alpha = 0.95 \)  
(f) \( n = 8, 1 - \alpha = 0.99 \)  

When confidence level is fixed, what change of \( E(L^2) \) do you observe when \( n \) changes? When sample size \( n \) is fixed, what change of \( E(L^2) \) do you observe when \( 1 - \alpha \) changes? Are these changes expected?

Hint: First express the length of the interval, \( L \), and then find the distribution of \( L \), then express \( E(L^2) \) in terms of \( \sigma^2 \). The answers to these questions are expression of \( \sigma^2 \).