Hypothesis Testing Based on Two Samples

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It is quite common to compare the properties of two distributions, for example, we would like to see which distribution has a higher mean, or which distribution has a higher variance, etc. This note describes how to conduct hypothesis testing regarding the mean and variance when the two distributions under consideration are normal.

1 Comparing the Means of Two Normal Distributions

Let us consider a problem in which random samples are available from two normal distributions. The problem is to determine whether the means of the two distributions are equal. Specifically, we assume that the random variables $X_1, \ldots, X_m$ form a random sample of size $m$ from a normal distribution for which both the mean $\mu_1$ and the variance $\sigma_1^2$ are unknown; and that the variables $Y_1, \ldots, Y_n$ form another independent random sample of size $n$ from another normal distribution for which both the mean $\mu_2$ and variance $\sigma_2^2$ are unknown. In this section, we will discuss several frequently seen cases.

1.1 The Case $\sigma_1^2 = \sigma_2^2 = \sigma^2$

We shall assume at this moment that the variance $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is the same for both distributions, even though the exact values are unknown.

Suppose it is desired to test the following hypotheses at a specified level of significance $\alpha$ ($0 < \alpha < 1$):

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_a : \mu_1 \neq \mu_2$$

Intuitively, it makes sense to reject $H_0$ if $\bar{X}_m - \bar{Y}_n$ is very different from zero, where $\bar{X}_m$ and $\bar{Y}_n$ are the means of the two samples, respectively. In spirit of the $t$ test, we define

$$S_X^2 = \sum_{i=1}^{m} (X_i - \bar{X}_m)^2 \quad \text{and} \quad S_Y^2 = \sum_{j=1}^{n} (Y_j - \bar{Y}_n)^2$$

Then the test statistic we shall use is

$$T = \frac{(m + n - 2)^{1/2}(\bar{X}_m - \bar{Y}_n)}{\left(\frac{1}{m} + \frac{1}{n}\right)^{1/2} (S_X^2 + S_Y^2)^{1/2}}$$
Next, let us derive the distribution of $T$.

For each pair of values $\mu_1$ and $\mu_2$, and for each $\sigma^2$, the sample mean $\bar{X}_m$ has a normal distribution with mean $\mu_1$ and variance $\sigma^2/m$, i.e.

\[
\bar{X}_m \sim N(\mu_1, \sigma^2/m)
\]

similarly

\[
\bar{Y}_n \sim N(\mu_2, \sigma^2/n)
\]

because both samples are from normal distribution. Furthermore, $\bar{X}_m$ and $\bar{Y}_n$ are independent. It follows that the difference $\bar{X}_m - \bar{Y}_n$ has a normal distribution with mean $\mu_1 - \mu_2$ and variance $[(1/m) + (1/n)]\sigma^2$, i.e.,

\[
\bar{X}_m - \bar{Y}_n \sim N(\mu_1 - \mu_2, [(1/m) + (1/n)]\sigma^2).
\]

Therefore, when the null hypothesis is true, i.e. $\mu_1 = \mu_2$, the following random variable $Z$ will have a standard normal distribution:

\[
Z = \frac{\bar{X}_m - \bar{Y}_n}{(1/m + 1/n)^{1/2}} \sim N(0, 1)
\]

Also, for all values of $\mu_1$, $\mu_2$, and $\sigma^2$, the random variable $S_X^2/\sigma^2$ has a $\chi^2$ distribution with $m - 1$ degrees of freedom, and $S_Y^2/\sigma^2$ has a $\chi^2$ distribution with $n - 1$ degrees of freedom, and the two random variables are independent. By the additive property of $\chi^2$ distribution, the following random variable $W$ has a $\chi^2$ distribution with $m + n - 2$ degrees of freedom:

\[
W = \frac{S_X^2}{\sigma^2} + \frac{S_Y^2}{\sigma^2} = \frac{S_X^2 + S_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}
\]

Furthermore, the four random variables $\bar{X}_m$, $\bar{Y}_n$, $S_X^2$, and $S_Y^2$ are independent. This is because: (i) $\bar{X}_m$ and $S_X^2$ are functions of $X_1, \cdots, X_m$; while $\bar{Y}_n$ and $S_Y^2$ are functions of $Y_1, \cdots, Y_n$; and we know that $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ are independent. Therefore $\{\bar{X}_m, S_X^2\}$ and $\{\bar{Y}_n, S_Y^2\}$ are independent. (ii) By the property of sample mean and sample variance, $\bar{X}_m$ and $S_X^2$ are independent, and $\bar{Y}_n$ and $S_Y^2$ are independent.

It follows that $Z$ are $W$ are independent. When the null hypothesis is true, i.e. $\mu_1 = \mu_2$, $Z \sim N(0, 1)$, and $W \sim \chi^2_{m+n-2}$. By the definition of $t$-distribution, we have the following

\[
T = \frac{(m + n - 2)^{1/2}(\bar{X}_m - \bar{Y}_n)}{(1/m + 1/n)^{1/2}(S_X^2 + S_Y^2)^{1/2}} = \frac{\bar{X}_m - \bar{Y}_n}{[S_X^2 + S_Y^2]/(m+n-2)^{1/2}} = \frac{Z}{[W/(m + n - 2)]^{1/2}} \sim t_{m+n-2}
\]

Thus, to test the hypotheses

\[
H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_a : \mu_1 \neq \mu_2,
\]
intuitively, if the null hypothesis was correct, the samples means \( \bar{X}_m \) and \( \bar{Y}_n \) should be very close. Therefore, we can use \( T \) as the test statistic, and the distribution of the statistic \( T \) is \( t_{m+n-2} \) when \( H_0 \) is correct.

Now suppose we want to test the hypotheses at the significant level \( \alpha \), that is under the null hypothesis \( \mu_1 = \mu_2 \), the probability of rejecting \( H_0 \) is \( \alpha \). As we analyzed before, we make our decision based on the difference of \( T \) from 0, that is if the value of \( |T| > c \), we will reject the null hypothesis, where \( c \) is a constant. Therefore the rejection probability is \( P(|T| > c) = \alpha \) where \( T \sim t_{m+n-2} \). It is easy to calculate that the value of \( c \) is \( 100 \times (1 - \alpha/2) \) percentile of \( t_{m+n-2} \) distribution, \( t_{m+n-2}(1 - \alpha/2) \). Therefore, the final decision rule is: if \( |T| > t_{m+n-2}(1 - \alpha/2) \), we would reject the null hypothesis at the significant level \( \alpha \); otherwise, we fail to reject the null hypothesis.

Now let us consider the one-sided test

\[ H_0 : \mu_1 \leq \mu_2 \quad \text{vs.} \quad H_a : \mu_1 > \mu_2. \]

In this case, we would reject the null hypothesis if the test statistic \( T \) is greater than a predetermined positive number \( c \). If the significant level is \( \alpha \), we can calculate the value of \( c \) by \( P(T > c) = \alpha \) where \( T \sim t_{m+n-2} \). Then \( c = t_{m+n-2}(1 - \alpha) \). Therefore if the alternative hypothesis is \( H_a : \mu_1 > \mu_2 \), we can reject the null hypothesis if the test statistic \( T > t_{m+n-2}(1 - \alpha) \).

The other case of one-sided hypothesis is

\[ H_0 : \mu_1 \geq \mu_2 \quad \text{vs.} \quad H_a : \mu_1 < \mu_2. \]

Similarly we would reject the null hypothesis if the test statistic \( T \) is less than a predetermined negative number \( c \). If the significant level is \( \alpha \), we can calculate the value of \( c \) by \( P(T < c) = \alpha \) where \( T \sim t_{m+n-2} \). Then \( c = t_{m+n-2}(\alpha) \). Therefore if the alternative hypothesis is \( H_a : \mu \leq \mu_0 \), we can reject the null hypothesis if the test statistic \( T < t_{m+n-2}(\alpha) \).

### 1.2 The Case \( \sigma^2_2 = k\sigma^2_1 \)

In the above, we discussed the case in which the two variances of the normal distributions are the same although unknown. Now, let us generalize a little bit, suppose know that \( \sigma^2_2 = k\sigma^2_1 \), where \( k \) is a known constant. With these assumptions, we want to test the hypotheses

\[ H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_a : \mu_1 \neq \mu_2. \]

For this problem, we still need a test statistic and the distribution of the statistic when the null hypothesis is true. However, the statistic \( T \) is not appropriate here because \( T \) assumes \( \sigma^2_1 = \sigma^2_2 \). However, we can define another statistic similar to \( T \) as following:

\[ T' = \frac{(m + n - 2)^{1/2}(\bar{X}_m - \bar{Y}_n)}{\left( \frac{1}{m} + \frac{k}{n} \right)^{1/2} (S^2_X + \frac{k S^2_Y}{n})^{1/2}} \]
Let us find out the distribution of $T'$ when the null hypothesis is true.

Similar as before,

$$X_m - Y_n \sim N(\mu_1 - \mu_2, \sigma_1^2/m + \sigma_2^2/n) = N(\mu_1 - \mu_2, [(1/m) + (k/n)]\sigma_1^2).$$

Therefore, when the null hypothesis is true, i.e. $\mu_1 = \mu_2$, the following random variable $Z$ will have a standard normal distribution:

$$Z = \frac{\bar{X}_m - \bar{Y}_n}{\left(\frac{1}{m} + \frac{k}{n}\right)^{1/2}\sigma_1} \sim N(0, 1)$$

Also, the random variable $S_X^2/\sigma_1^2$ has a $\chi^2$ distribution with $m - 1$ degrees of freedom, and $S_Y^2/\sigma_2^2$ has a $\chi^2$ distribution with $n - 1$ degrees of freedom, and the two random variables are independent. By the additive property of $\chi^2$ distribution, the following random variable $W$ has a $\chi^2$ distribution with $m + n - 2$ degrees of freedom:

$$W = \frac{S_X^2}{\sigma_1^2} + \frac{S_Y^2}{\sigma_2^2} = \frac{S_X^2 + S_Y^2/k}{\sigma_1^2} \sim \chi^2_{m+n-2}$$

$Z$ and $W$ are independent due to the same reason as before. By the definition of $t$-distribution, we have

$$T' = \frac{(m + n - 2)^{1/2}(\bar{X}_m - \bar{Y}_n)}{\left(\frac{1}{m} + \frac{k}{n}\right)^{1/2}\sigma_1} = \frac{\bar{X}_m - \bar{Y}_n}{\left(\frac{1}{m} + \frac{k}{n}\right)^{1/2}\sigma_1} = \frac{Z}{\left[\frac{W}{(m + n - 2)^{1/2}}\right]^{1/2}} \sim t_{m+n-2}$$

As we can see, if $k = 1$, the statistic $T'$ will reduce to $T$. After we figure out the test statistic $T'$ and the null distribution, we can design a testing procedure as before.

### 1.3 Test $H_0 : \mu_1 - \mu_2 = \lambda$ vs. $H_a : \mu_1 - \mu_2 \neq \lambda$

In the above, we only checked the two means are equal or not. We can generalize the above discussion a step further, i.e. to test

$$H_0 : \mu_1 - \mu_2 = \lambda \quad \text{vs.} \quad H_a : \mu_1 - \mu_2 \neq \lambda$$

where $-\infty < \lambda < \infty$ is a constant. As we can see, if $\lambda = 0$, this test reduces to the test we discussed above.

To test these hypotheses, similar to the above discussion, when the variances of the two normal distributions are equal, we can define the test statistic as

$$T = \frac{(m + n - 2)^{1/2}(\bar{X}_m - \bar{Y}_n - \lambda)}{\left(\frac{1}{m} + \frac{1}{n}\right)^{1/2}(S_X^2 + S_Y^2)^{1/2}}.$$
Similarly, it can be shown that when the null hypothesis $H_0$ is true, $T \sim t_{m+n-2}$. We can make our decision based on the value of the test statistic and the null distribution $t_{m+n-2}$.

When the variances of the two normal distributions are not equal, but their relation is known as $\sigma_2^2 = k\sigma_1^2$, we can define the test statistic as

$$T' = \frac{(m+n-2)^{1/2}(\bar{X}_m - \bar{Y}_n - \lambda)}{\left(\frac{1}{m} + \frac{k}{n}\right)^{1/2}(S_X^2 + S_Y^2/k)^{1/2}}.$$  

It can be shown that when the null hypothesis $H_0$ is true, $T' \sim t_{m+n-2}$. We can make our decision based on the value of the test statistic and the null distribution $t_{m+n-2}$.

Similarly, we can define the test procedure for the one-sided hypotheses.

1.4 Confidence Intervals for $\mu_1 - \mu_2$

As we studied before, the confidence intervals and the hypothesis testing process are equivalent. Therefore, from the testing process, we can construct the $1 - \alpha$ confidence intervals for the difference of means of two normal distributions.

Firstly, under the case, $\sigma_1^2 = \sigma_2^2$, the $1 - \alpha$ confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{X}_m - \bar{Y}_n) \pm t\left(1 - \frac{\alpha}{2}, m+n-2\right)\sqrt{\frac{S_X^2 + S_Y^2}{m + n - 2}}\sqrt{\frac{1}{m} + \frac{1}{n}}.$$  

Indeed, in this case

$$h(X_1, \ldots, X_m, Y_1, \ldots, Y_n, \mu_1, \mu_2) = \frac{(m+n-2)^{1/2}[(\bar{X}_m - \bar{Y}_n) - (\mu_1 - \mu_2)]}{\left(\frac{1}{m} + \frac{1}{n}\right)^{1/2}(S_X^2 + S_Y^2)^{1/2}}$$

is the pivot to construct the confidence interval.

Similarly, under the case, $\sigma_2^2 = k\sigma_1^2$, the $1 - \alpha$ confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{X}_m - \bar{Y}_n) \pm t\left(1 - \frac{\alpha}{2}, m+n-2\right)\sqrt{\frac{S_X^2 + S_Y^2/k}{m + n - 2}}\sqrt{\frac{1}{m} + \frac{k}{n}}.$$  

and, in this case

$$h(X_1, \ldots, X_m, Y_1, \ldots, Y_n, \mu_1, \mu_2) = \frac{(m+n-2)^{1/2}[(\bar{X}_m - \bar{Y}_n) - (\mu_1 - \mu_2)]}{\left(\frac{1}{m} + \frac{k}{n}\right)^{1/2}(S_X^2 + S_Y^2/k)^{1/2}}$$

is the pivot to construct the confidence interval.
2 Comparing the Variances of Two Normal Distributions

Assume that the random variables $X_1, \cdots, X_m$ form a random sample of size $m$ from a normal distribution for which both the mean $\mu_1$ and the variance $\sigma_1^2$ are unknown; and that the variables $Y_1, \cdots, Y_n$ form another independent random sample of size $n$ from another normal distribution for which both the mean $\mu_2$ and variance $\sigma_2^2$ are unknown. In this section, we will develop testing procedure to compare the two variances.

2.1 $F$ Distribution

We first define and discuss the properties of a probability distribution family, called the $F$ distribution (Fisher-Snedecor distribution). This distribution arises in many important problems of testing hypotheses in which two or more normal distributions are to be compared on the basis of random samples from each of the distribution.

Consider two independent random variables $Y$ and $W$, such that $Y$ has a $\chi^2$ distribution with $m$ degrees of freedom, and $W$ has a $\chi^2$ distribution with $n$ degrees of freedom, where $m$ and $n$ are given positive integers. Define a new random variable $X$ as follows:

$$X = \frac{W}{m} \frac{W}{n} = \frac{W}{m} \frac{n}{m}.$$

Then the distribution of $X$ is called an $F$ distribution with $m$ and $n$ degrees of freedom.

It can be proved that the if a random variable $X$ has an $F$ distribution with $m$ and $n$ degrees of freedom, i.e. $X \sim F(m, n)$, then its p.d.f. is as follows:

$$f(x) = \left\{ \begin{array}{ll}
\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{x^{(m/2)-1}}{(m+n)/(m+n)^{m+n}/2}, & x > 0 \\
0, & x \leq 0
\end{array} \right.$$

Please note that when we speak of an $F$ distribution with $m$ and $n$ degrees of freedom, the order in which the numbers $m$ and $n$ are given is important. As we can see from the definition or the p.d.f. of $X$, when $m \neq n$, the $F$ distribution with $m$ and $n$ degrees of freedom and the $F$ distribution with $n$ and $m$ degrees of freedom are two different distributions. In fact, from its definition, it is straightforward to see that if $X \sim F(m, n)$, then its reciprocal $1/X$ will have an $F(n, m)$ distribution.

From the definitions of $t$ distribution and $F$ distribution, it is easy to see that if $X \sim t(n)$, then $X^2 \sim F(1, n)$. Indeed, since $X \sim t(n)$, then $X$ could be written as $X = \frac{Z}{\sqrt{Y/n}}$, where $Z \sim N(0, 1)$ and $Y \sim \chi^2_n$. Then $X^2 = \frac{Z^2}{Y/n} = \frac{Z^2}{Y/n}$, where $2Z = Z^2 \sim \chi^2_1$, by the definition of $F$ random variable, $X^2 \sim F(1, n)$. 


2.2 Comparing the Variances of Two Normal Distributions

Now let us go back to the problem of comparing the variances of the two normal distributions proposed at the beginning of this section. Suppose that finally the following hypotheses are to be tested at given level of significance $\alpha$,

$$H_0 : \sigma^2_1 = \sigma^2_2 \quad \text{vs.} \quad H_a : \sigma^2_1 \neq \sigma^2_2.$$ 

As in the previous section, we define

$$S^2_X = \frac{1}{m} \sum_{i=1}^{m} (X_i - \bar{X}_m)^2 \quad \text{and} \quad S^2_Y = \frac{1}{n} \sum_{j=1}^{n} (Y_j - \bar{Y}_n)^2.$$ 

As proved before, $S^2_X/(m-1)$ and $S^2_Y/(n-1)$ are unbiased estimators of $\sigma^2_1$ and $\sigma^2_2$, respectively. Intuitively, it makes sense that if the null hypothesis was true, the ratio

$$V = \frac{S^2_X/(m-1)}{S^2_Y/(n-1)}$$

should be close to 1. Therefore, we would reject the null hypothesis if the test statistic $V$ is too small or too big.

Under the null hypothesis, $\sigma^2_1 = \sigma^2_2$,

$$V = \frac{S^2_X/(m-1)}{S^2_Y/(n-1)} = \frac{[S^2_X/\sigma^2_1]/(m-1)}{[S^2_Y/\sigma^2_2]/(n-1)}.$$ 

It is well-known that

$$\frac{S^2_X}{\sigma^2_1} \sim \chi^2_{m-1} \quad \text{and} \quad \frac{S^2_Y}{\sigma^2_2} \sim \chi^2_{n-1},$$

by the definition of $F$ distribution, $V \sim F(m-1, n-1)$.

By our intuitive decision rule, we would reject $H_0$ either $V \leq c_1$ or $V \geq c_2$, where $c_1$ and $c_2$ are two constants such that $P(V \leq c_1) + P(V \geq c_2) = \alpha$. The most convenient choice of $c_1$ and $c_2$ is the one that makes $P(V \leq c_1) = P(V \geq c_2) = \alpha/2$. That is, choose $c_1 = F(\alpha/2, m-1, n-1)$, the $\alpha/2$ percentile of $F(m-1, n-1)$, and choose $c_2 = F(1 - \alpha/2, m-1, n-1)$, the $1 - \alpha/2$ percentile of $F(m-1, n-1)$.

This procedure can also be generalized to test the hypotheses

$$H_0 : \sigma^2_1 = r \sigma^2_2 \quad \text{vs.} \quad H_a : \sigma^2_1 \neq r \sigma^2_2,$$

where $r$ is a positive constant, and we notice that when $r = 1$ this will be reduced to the previous case. Under this new null hypothesis, the value of $V$ would be close to $r$, therefore, $V/r$ would be close to 1. Let us have a close look at the statistic $V/r$:

$$\frac{V}{r} = \frac{S^2_X/(m-1)}{r S^2_Y/(n-1)} = \frac{\frac{S^2_X}{\sigma^2_1}}{r \frac{S^2_Y}{\sigma^2_1}} / (m-1) = \frac{[S^2_X/\sigma^2_1]/(m-1)}{[S^2_Y/\sigma^2_2]/(n-1)}.$$
It follows that the statistic $V/r \sim F(m - 1, n - 1)$ for the same reason as above.

With the test statistic $V/r$ and the sampling distribution of the statistic under $H_0$, we would reject the null hypothesis at the significant level $\alpha$ if $V/r \leq c_1$ or $V/r \geq c_2$, with $c_1 = F(\alpha/2, m - 1, n - 1)$ and $c_2 = F(1 - \alpha/2, m - 1, n - 1)$.

Now, let us consider the one-sided hypotheses

$$H_0 : \sigma_1^2 \leq \sigma_2^2 \quad \text{vs.} \quad H_a : \sigma_1^2 > \sigma_2^2.$$ 

As before, we use the ratio

$$V = \frac{S_X^2/(m - 1)}{S_Y^2/(n - 1)}.$$

Since $S_X^2/(m - 1)$ and $S_Y^2/(n - 1)$ are unbiased estimators of $\sigma_1^2$ and $\sigma_2^2$, respectively. The data evidence prefers $H_a$ if the ratio $V$ is significantly big. Given a significant level $\alpha$, we would reject the null hypothesis $H_0$ if $V > c$ where $c$ is a constant, and we choose $c = F(1 - \alpha, m - 1, n - 1)$.

### 2.3 Confidence Intervals for the Ratio $\sigma_1^2/\sigma_2^2$

As we studied before, the confidence intervals and the hypothesis testing process are equivalent. Therefore, from the testing process, we can construct the $1 - \alpha$ confidence intervals for the ratio of variances of two normal distributions.

Let $r = \sigma_1^2/\sigma_2^2$, then equivalently, we are looking for a $1 - \alpha$ confidence intervals for $r$. As proved in last subsection, we know $V/r \sim F(m - 1, n - 1)$, therefore we can design the pivot as

$$h(X_1, \cdots, X_m, Y_1, \cdots, Y_n, r) = \frac{V}{r} = \frac{S_X^2/(m - 1)}{rS_Y^2/(n - 1)} \sim F(m - 1, n - 1).$$

Choose $c_1 = F(\alpha/2, m - 1, n - 1)$ and $c_2 = F(1 - \alpha/2, m - 1, n - 1)$, then

$$P(c_1 \leq V/r \leq c_2) = 1 - \alpha,$$

from which we can obtain the desired confidence interval as

$$\frac{V}{F(1 - \alpha/2, m - 1, n - 1)} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{V}{F(\alpha/2, m - 1, n - 1)}.$$

### 3 Exercises

**Exercise 1.** Assume that the random variables $X_1, \cdots, X_m$ form a random sample of size $m$ from a normal distribution for which the mean $\mu_1$ is unknown and the variance $\sigma_1^2$ is known;
and that the variables $Y_1, \cdots, Y_n$ form another independent random sample of size $n$ from another normal distribution for which the mean $\mu_2$ is unknown and variance $\sigma_2^2$ is known. Please develop and testing procedure to test the following hypotheses at the significance level $\alpha$:

$$H_0 : \mu_1 = \mu_2 \quad vs. \quad H_a : \mu_1 \neq \mu_2.$$  

Using the relation between hypothesis testing and confidence interval, identify the $1 - \alpha$ confidence interval for $\mu_1 - \mu_2$.

**Exercise 2.** Let $X_1, \cdots, X_m$ be a random sample from the exponential density with mean $\theta_1$, and let $Y_1, \cdots, Y_n$ be a random sample from the exponential density with mean $\theta_2$. The density function for exponential distribution is

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}; \quad x > 0$$

**a.** Define an $F$-test procedure test the hypotheses

$$H_0 : \theta_1 = \theta_2 \quad vs. \quad H_a : \theta_1 \neq \theta_2.$$  

[Hint: Find the connection between $\sum_{i=1}^{m} X_i$ and a $\chi^2$ random variable. Do the same thing to $\sum_{i=1}^{n} Y_i$]

**b.** Generalize the above testing procedure to test the hypotheses

$$H_0 : \theta_1 = r\theta_2 \quad vs. \quad H_a : \theta_1 \neq r\theta_2,$$

where $r$ is a positive number.

**c.** Find a $1 - \alpha$ confidence interval for $\theta_1/\theta_2$. 